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ELASTIC COLUMNS UNDER HALF-SINE PULSE LOADING

LAWRENCE H. TAYLOR, JR.

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ELASTIC COLUMNS UNDER HALF-SINE PULSE LOADING

* * * * *

Lawrence H. Taylor, Jr.

ELASTIC COLUMNS UNDER
HALF-SINE PULSE LOADING

by

Lawrence H. Taylor Jr.
//
Lieutenant, United States Navy

Submitted in partial fulfillment of
the requirements for the degree of

MASTER OF SCIENCE
IN
MECHANICAL ENGINEERING

United States Naval Postgraduate School
Monterey, California

1962

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ELASTIC COLUMNS UNDER HALF-SINE PULSE LOADING

by

Lawrence H. Taylor Jr.

This work is accepted as fulfilling
the thesis requirements for the degree of
MASTER OF SCIENCE
IN
MECHANICAL ENGINEERING
from the
United States Naval Postgraduate School

ABSTRACT

This paper describes an investigation of the dynamic behavior of a pin ended elastic column, subjected to half-sine pulse loading applied with small eccentricity. The column is replaced by a lumped parameter mathematical model, and the equations for the model are solved with a high speed digital computer. The failure criterion used is a limiting value of extreme fiber strain. The minimum loads which cause failure are found as a function of load duration for columns having the slenderness ratios 50, 100, and 150. It is shown that an elastic column can support rapidly applied dynamic loads greatly in excess of the Euler load. As the duration of the load pulse is decreased, the lateral deflection at failure becomes progressively smaller and the effects of axial inertia become increasingly significant.

ACKNOWLEDGMENT

The author would like to express his appreciation for the guidance and encouragement given him by Professor R. E. Newton, Chairman of the Mechanical Engineering Department of the U. S. Naval Postgraduate School, throughout the course of this investigation. The very helpful suggestions received from Professor J. E. Brock are also appreciated.

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TABLE OF SYMBOLS

Symbol	Description	Dimensions
A	Area of the column cross section	L^2
A_i	dimensionless x direction component of force transmitted between mass i -1 and mass i ($= P_i/F$)	-
B_i	dimensionless y direction component of force transmitted between mass i -1 and mass i ($= V_i/F$)	-
C	bending stiffness of one of the length increments of the mathematical model ($= EI/\ell$)	FL
c	distance from the centroid to the extreme fiber	L
E	Young's modulus for the column material	FL^{-2}
e	eccentricity of load application	L
F	Euler load ÷ π^2 ($= EI/L^2 = P_E/\pi^2$)	F
I	rectangular moment of inertia of the column cross section	L^4
K	axial stiffness of one of the length increments of the mathematical model ($= EA/\ell$)	FL^{-1}
L	overall length of the column	L
ℓ	length of one length increment of the mathematical model ($= L/n$)	L
M_i	bending moment at mass i	FL
$\mathfrak{M}_i$	dimensionless bending moment at mass i ($= M_i/F\ell$)	-
m_i	mass of mass i of the mathematical model	M
n	number of length increments in the mathematical model	-
P_i	x direction component of force transmitted between mass i - 1 and mass i	F
P_E	Euler load for the column ($= \pi^2 EI/L^2$)	F

Symbol	Description	Dimensions
r	radius of gyration of the column cross section ($= \sqrt{I/A}$)	L
S	slenderness ratio of the column ($= L/r$)	-
s	velocity of travel of an elastic wave in the column material ($= \sqrt{E/\gamma}$)	LT^{-1}
T	dimensionless time ($= t/\tau_1$)	-
t	real time	T
u_i	displacement of mass i in the x direction	L
V_i	y direction component of force transmitted between mass i - 1 and mass i	F
v_i	displacement of mass i in the y direction	L
x	cartesian coordinate in the direction of the undeflected centroidal axis of the column	-
y	cartesian coordinate in the direction in which lateral deflection of the column occurs	-
α	ratio of the time required for an elastic wave to travel one length increment to the time increment in the computer program ($= \ell/s \Delta t$)	-
β	ratio of the natural period of first mode lateral vibration of the column to the half period of the sine wave force pulse ($= \tau_1 / \frac{\tau_f}{2}$)	-
γ	density of the column material	ML^{-3}
δ	ratio of the natural period of first mode lateral vibration of the column to the terminate time of the computer program	-
ϵ	longitudinal strain	
θ_i	angle in radians between the length increment from mass i - 1 to mass i and the x direction	-
μ	mass per unit length of the column	ML^{-1}
ρ	slenderness ratio of one of the length increments of the mathematical model ($= \ell/r$)	-

τ_1	natural period of first mode lateral vibration of the column	T
τ_f	period of the sine wave of the force pulse	T
Φ	tolerance in tabulated failure values of P_1/P_E	-
ω_f	circular frequency of the sine wave of the force pulse	T^{-1}
ΔT	dimensionless time increment used in the numerical solution ($= \Delta t/\tau_1$)	-
$\Delta \ell_i$	change in length of length increment between mass i - 1 and mass i	L
ΔM_i	change in bending moment between mass i - 1 and mass i ($= M_i - M_{i-1}$)	FL
Δu_i	difference between longitudinal deflection of mass i and mass i - 1 ($= u_i - u_{i-1}$)	L
Δv_i	difference between lateral deflection of mass i and mass i - 1 ($= v_i - v_{i-1}$)	L

1. Introduction

The problems of the dynamic behavior of structures are ones which, like so many others, have only recently been attacked with any sort of vigor by the engineering profession. Whether this is due to the previous unimportance of the problems or to the lack of tools for their solution is primarily of historical interest; the fact remains that the problems are of importance today, and tools are now available for the solutions.

One of these problems of structural dynamics which has aroused considerable interest in recent years is that of a dynamically loaded column. This seemingly simple structural member, whose behavior under conditions of static loading was predicted by Euler in 1757, becomes a quite complex system when the loading is applied dynamically.

Attacks on this problem have been concentrated thus far on solutions for two types of loading - constant velocity loading of one end of the column, and impact loading. Hoff [1,2] [1] has treated the case of an elastic column, initially curved in the shape of a half sine wave, subjected to constant velocity loading such as that encountered during compression tests in commercial testing machines. He has shown that rapidly loaded slender columns with small initial deflections will support loads greatly in excess of the Euler load. Chawla [3] has extended this work to include the case of inelastic columns. Sevin [4] has confirmed Hoff's results, while retaining the effects of axial inertia (which were not considered by Hoff), and one of his conclusions is that

> ...so long as the column remains elastic, axial inertia effects are of negligible importance in so far as the gross behavior of

---

[1] Numbers in brackets refer to bibliography.

conventional structural columns is concerned regardless of the initial deflected shape, end fixity, or type of axial loading.

Gerard and Becker [5] have studied the impact loading case using the unloading strain wave produced upon failure of a tension specimen, and have concluded that a column may momentarily withstand any magnitude of compressive stress, and that the buckling may occur over a small portion of the length of the column, rather than the entire length.

The literature also contains several treatments [6,7,8] of the problem of the stability of a pin ended column subjected to an axial force of the form $P_1 = P_o + A \sin \omega t$. It has been shown that, for certain values of ω, the maximum compressive force $P_o + A$ may become much higher than the Euler load without causing instability. On the other hand, it is also found that instability may exist when P_o is a tensile force, provided that A and ω have the proper values.

Konig and Taub [9] have treated the case of a pin ended column with an initial half sine deflection, subject to a suddenly applied force of constant magnitude and variable duration. Their investigation shows that perfectly elastic columns can support loads in excess of the Euler load when the duration of the load is short.

Other than in this last reference, the problem of the prediction of the load carrying capacity of a dynamically loaded column subjected to an externally applied force pulse of specified shape and duration seems to have been neglected. This is the problem which is considered in the present investigation.

The column is assumed to be perfectly elastic, and is initially straight, rather than having some initial curved shape. It is of constant cross section, has constant physical properties, and is free of

any damping. The effects of rotary inertia and shear strains are neglected, but axial inertia effects are retained, as well as non-linear axial strain components due to bending.

The loading imposed on the column is a half sine pulse of force, applied with an arbitrarily chosen eccentricity, in an axial direction. The column is considered to have hinged-hinged end conditions, with the loaded end free to translate in the longitudinal direction. The unloaded end has an eccentric fixed-pin connection which allows only rotation.

Since the concept of stability or buckling of the column seems to lose its meaning when applied to columns subjected to dynamic loading, some other criterion of failure must be used. In this study, an arbitrarily selected value of the extreme fiber strain is used to define failure.

The problem is formulated and solved, not in terms of the real column, but in terms of a lumped parameter type of mathematical model. A set of equations is developed for this model and is then solved in a high speed digital computer.

---

2. The Mathematical Model and Development of the Equations for the System.

In order to study the dynamic behavior of the column, the real column is replaced by a lumped parameter model consisting of a series of hinged rods, with point masses at the hinged joints. A general section of the mathematical model is shown in Fig. 1. The complete model consists of n increments, each of length $\ell = L/n$, and n + 1 point masses, each having a mass of $m = \mu \ell$ (except those at either end, whose masses are $\mu \ell/2$). The model is initially perfectly straight, with an eccentric fixed pin connection at mass n + 1. The external load is applied eccentrically to mass number one, which is free to rotate and to move in the x direction,

but is restrained from motion in the y direction. The eccentricity of the fixed pin connection at mass number n + 1 is the same as that with which the loading is applied.

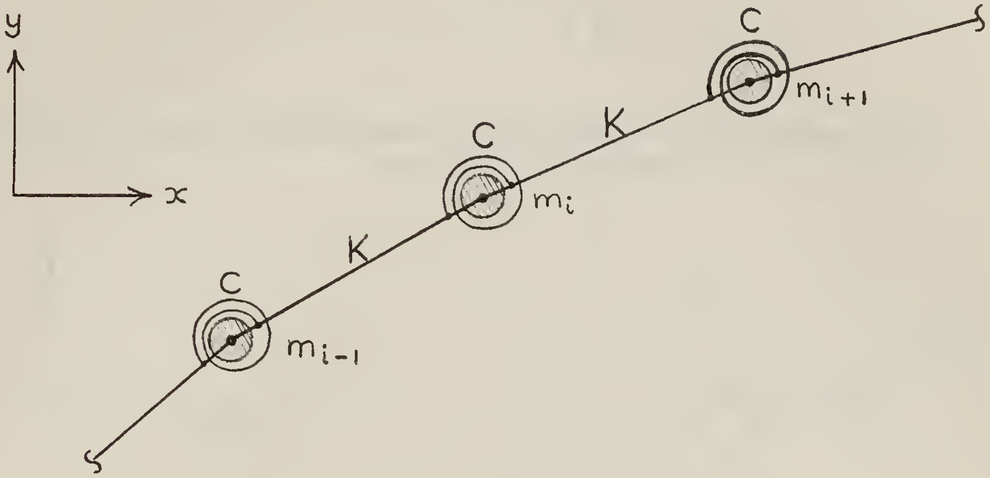


Figure 1. A section of the mathematical model of the column.

The masses are connected by perfectly elastic, massless rods, which are hinged to each other at the point masses, as shown. The rods have an axial stiffness of $K = EA/\ell$, where E is Young's modulus for the column material, and A is the area of the cross section of the column. The rods are considered flexurally rigid.

At each of the point masses, with the exceptions of those at either end, is a perfectly elastic, massless spiral spring, with a spring constant $C = EI/\ell$ for either direction of rotation. (I is the rectangular moment of inertia of the cross section of the column)

Referring now to the free body diagrams shown in Fig. 2 and Fig. 3, the following equations may be written:

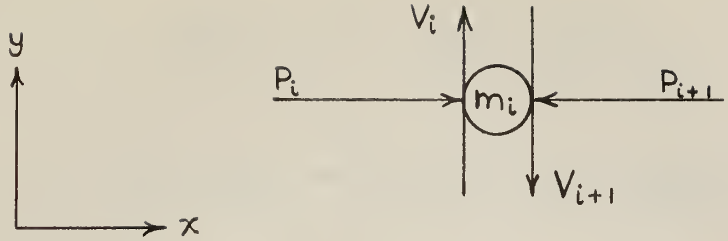


Fig. 2 Free body diagram of one mass of the mathematical model of the column.

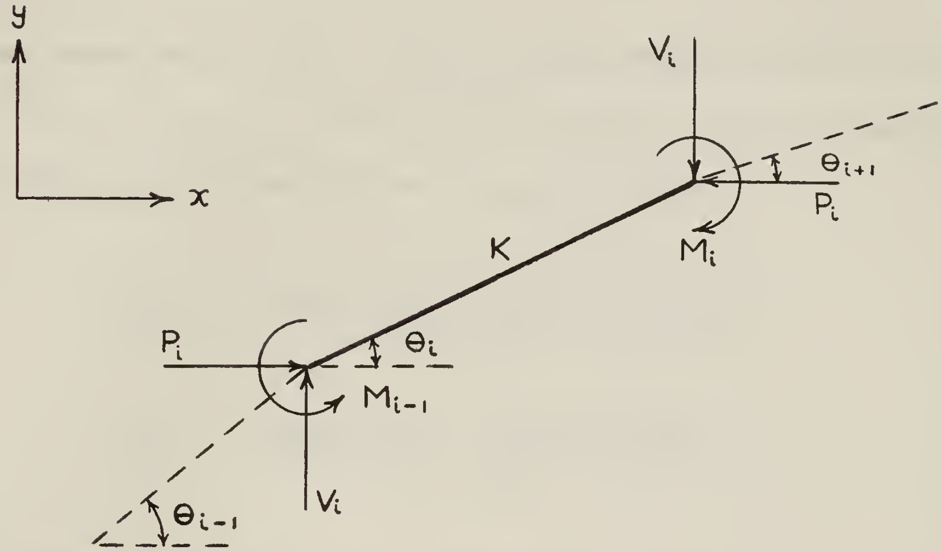


Fig. 3 Free body diagram of one of the length increments (rods) of the mathematical model of the column.

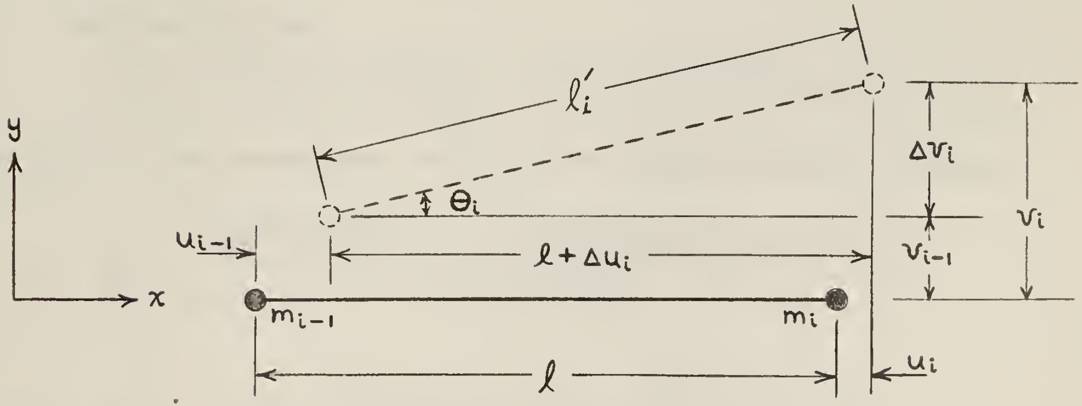


Fig. 4 The geometry of one of the length increments of the mathematical model of the column.

$$d^2u_i/dt^2 = (P_i - P_{i+1})/m_i \tag{1}$$

$$d^2v_i/dt^2 = (V_i - V_{i+1})/m_i \tag{2}$$

$$P_i \cos\theta_i + V_i \sin\theta_i = K\Delta\ell_i \tag{3}$$

$$\Delta M_i \equiv M_i - M_{i-1} = P_i\,\Delta v_i - V_i(\ell + \Delta u_i) \tag{4}$$

In these equations, $\Delta\ell_i = \ell - \ell_i'$ (see Fig. 4), $\Delta u_i = u_i - u_{i-1}$, $\Delta v_i = v_i - v_{i-1}$, $M_i = C(\theta_i - \theta_{i+1})$, and t is real time.

Solving eqs. (3) and (4) for P_i and V_i yields

$$P_i = \frac{\Delta M_i \sin\theta_i + K\Delta\ell_i(\ell + \Delta u_i)}{\ell - \Delta\ell_i} \tag{5}$$

$$V_i = \frac{K\Delta\ell_i\,\Delta v_i - \Delta M_i \cos\theta_i}{\ell - \Delta\ell_i} \tag{6}$$

In order to put these into dimensionless form, both sides of eqs. (5) and (6) are now divided by $F = P_E/\pi^2$, where P_E is the Euler load for the column. The numerator and denominator of both eqs. are divided by ℓ, while noting that

$$K/F = (EA/\ell)\cdot(L^2/EI) = (EA/\ell)\cdot(n^2\rho^2/EA) = n^2\rho^2/\ell$$

where $\rho = \frac{\ell}{r}$ is the slenderness ratio of the length increment, giving

$$A_i \equiv \frac{P_i}{F} = \frac{(\Delta M_i/F\ell)\sin\theta_i + n^2\rho^2(\Delta\ell_i/\ell)(1 + \Delta u_i/\ell)}{1 - \Delta\ell_i/\ell}$$

and

$$B_i \equiv \frac{V_i}{F} = \frac{n^2\rho^2(\Delta\ell_i/\ell)(\Delta v_i/\ell) - (\Delta M_i/F\ell)\cos\theta_i}{1 - \Delta\ell_i/\ell}$$

The following approximations are now made:

$$\sin\theta_i \approx \theta_i$$

$$\cos\theta_i \approx 1 - \theta_i^2/2$$

$$1 - \Delta\ell_i/\ell \approx 1$$

$$\Delta v_i/\ell \approx \theta_i$$

These approximations are in error by one per cent or less for the values of θ_i, $\Delta\ell_i$, Δu_i, and Δv_i anticipated.

Substituting in the equations for A_i and B_i, and using the notation $\Delta m_i = \Delta M_i/F\ell$, leads to

$$A_i = \Delta m_i\theta_i + n^2\rho^2(\Delta\ell_i/\ell)(1 + \Delta u_i/\ell)$$

and

$$B_i = n^2\rho^2(\Delta\ell_i/\ell)\theta_i - \Delta m_i(1 - \theta_i^2/2)$$

Now, reference to the geometry of Fig. 4 shows that

$$\ell' = (\ell + \Delta u_i)\cos\theta_i + \Delta v_i\sin\theta_i$$

or

$$\Delta\ell_i = \ell - \ell' = \ell - (\ell + \Delta u_i)\cos\theta_i - \Delta v_i\sin\theta_i$$

whence

$$\Delta\ell_i/\ell = 1 - (1 + \Delta u_i/\ell)\cos\theta_i - (\Delta v_i/\ell)\sin\theta_i$$

If the same approximations as were used previously are now substituted, this becomes

$$\Delta\ell_i/\ell = (\theta_i^2/2)(\Delta u_i/\ell) - (\Delta u_i/\ell + \theta_i^2/2)$$

We further assume that the product $(\theta_i^2/2)(\Delta u_i/\ell)$ is negligible compared to the sum $\Delta u_i/\ell + \theta_i^2/2$, so that

$$\Delta\ell_i/\ell \approx -(\Delta u_i/\ell + \theta_i^2/2)$$

This approximation is now substituted into the expressions for A_i and B_i, giving

$$A_i = \Delta m_i\theta_i - n^2\rho^2(\Delta u_i/\ell + \theta_i^2/2)(1 + \Delta u_i/\ell)$$

and

$$B_i = -n^2\rho^2(\Delta u_i/\ell + \theta_i^2/2) - \Delta m_i(1 - \theta_i^2/2)$$

These equations may be further reduced by assuming that

(a) $|\Delta u_i/\ell| \ll 1$

(b) $|\Delta m_i \theta_i| \ll |n^2\rho^2(\Delta u_i/\ell + \theta_i^2/2)|$

(c) $\theta_i^2/2 \ll 1$

The physical significance of assumptions (a) and (c) is clear. The assumption made in (b) is equivalent to neglecting the contribution of the y direction force component to the total force transmitted by the rod, i.e., omitting $V_i \sin \theta_i$ in eq. (3).

Based on the above,

$$A_i = -n^2\rho^2(\Delta u_i/\ell + \theta_i^2/2) \quad (7)$$

and

$$B_i = \theta_i A_i - \Delta m_i \quad (8)$$

Equations (7) and (8) are the dimensionless expressions for the components of the force transmitted by the ith length increment in the x and y directions, and are used in this form in the numerical solution.

Using the notation $\Delta P_i = P_i - P_{i+1}$ and $\Delta A_i = \Delta P_i/F$, equation (1) may be written as

$$d^2u_i/dt^2 = \Delta A_i/(m_i/F)$$

But, $m_i = \mu\ell = \gamma A\ell$ and $F = EA/n^2\rho^2$, so that

$$d^2u_i/dt^2 = \Delta A_i E/\gamma\ell n^2\rho^2$$

However, $E/\gamma = s^2$, where s is the velocity of travel of an elastic wave in the column material, so that

$$d^2u_i/dt^2 = \Delta A_i(s^2/\ell n^2\rho^2) = \Delta A_i(s^2/\ell^2)(\ell/n^2\rho^2) \quad (9)$$

and

These equations may be further reduced by assuming that

(a)

(b)

(c)

The physical significance of assumptions (a) and (c) is clear. The assumption made in (b) is equivalent to neglecting the contribution of the y direction force component to the total force transmitted by the rod.

Based on the above,

(7)

and

(8)

Equations (7) and (8) are the dimensionless expressions components of the force transmitted by the ith length increment for the in the x and y directions, and are used in this form in the numerical solution.

Using the notation and , equation (1) may be written as

But, and , so that

However, , where is the velocity of travel of an elastic wave in the column material, so that

(9)

For a beam with hinged ends the natural period of first mode vibration, τ_1, may be found from the expression

$$2\pi/\tau_1 = \pi^2 \left(EI/\mu L_t^4 \right)^{1/2}$$

Solving this for τ_1 gives

$$\tau_1 = 2\ell n^2 \rho/\pi s$$

which leads to

$$s^2/\ell^2 = 4n^4\rho^2/\pi^2 \tau_1^2$$

Substitution of this expression in (9) yields

$$d^2u_i/dt^2 = \Delta A_i \left(4n^2\ell/\pi^2 \tau_1^2 \right)$$

Or, since ℓ is not a function of time,

$$d^2(u_i/\ell)/dt^2 = \Delta A_i \left(4n^2/\pi^2 \tau_1^2 \right)$$

Dimensionless time is now defined as $T = t/\tau_1$, so that $(dt)^2 = \tau_1^2(dT)^2$. Therefore,

$$d^2(u_i/\ell)/dT^2 = \Delta A_i \left(2n/\pi\right)^2, \quad 1 \leq i \leq n \qquad (10)$$

By the same reasoning as has just been applied to eq. (1), eq. (2) may be reduced to

$$d^2(v_i/\ell)/dT^2 = \Delta B_i \left(2n/\pi\right)^2, \quad 1 \leq i \leq n \qquad (11)$$

Equations (10) and (11) are the equations for the dimensionless accelerations in the x and y directions respectively, for $1 \leq i \leq n$, which are used in the numerical solution. For the first mass, the acceleration in the x direction is twice the value given by eq. (10), due to this being a half mass.

In order to "solve" the acceleration equations for the n + 1 masses, it is necessary to specify the forcing function, boundary conditions, and initial conditions. These are specified as follows:

(a) for $0 \leq T \leq 1/\beta$, the external load applied (eccentrically) to mass number one is given by

$A_1 = A_0 \sin(\pi \beta T)$, where $\beta \equiv \tau_1 \Big/ \frac{\tau_f}{2}$

For $T > 1/\beta$, $A_1 = 0$. Specifying the applied force in this manner allows a selection of both the amplitude and duration of the pulse. The bending moment at mass one is given by $M_1 = P_1 e$, while that at mass n + 1 is $P_{n+1} e$.

(b) for $0 \leq T \leq \infty$; $v_1/\ell = 0$, $v_{n+1}/\ell = 0$, $u_{n+1}/\ell = 0$

(c) at T = 0, for $1 \leq i \leq n + 1$:

$$v_i/\ell = 0 \ , \quad u_i/\ell = 0 \ , \quad d(v_i/\ell)/dT = 0, \quad d(u_i/\ell)/dT = 0$$

The choice of the eccentricity with which the force is applied is made arbitrarily. The value used, given in dimensionless form, is e/r = 0.05, where e is the actual eccentricity, and r is the radius of gyration of the column cross section.

Another arbitrary choice which is made is that of a failure criterion for the column. For this investigation, the column is said to have failed when the extreme fiber strain, ϵ_{max}, reaches a value of 0.01. This represents a stress of 300,000 psi in steel, which is admittedly high, but which is also an attainable yield point for certain alloy steels. It is felt that the use of this high value sets an upper limit for columns fabricated from presently available material.

In order to compute the value of the extreme fiber strain, ϵ_{max}, it is necessary to find both the centroidal axis strain ϵ_c and the strain caused by the application of bending moments, since $\epsilon_{max} = |\epsilon_c| + |\epsilon_b|$. The centroidal axis strain in any length increment of the model is calculated from

$$\epsilon_{ci} = \Delta\ell_i/\ell = -\left(\Delta u_i/\ell + \theta_i^2/2\right)$$

The strain in the same length increment, caused by the bending moments, may be computed as

$$\epsilon_{bi} = (c M_{avg.})/EI$$

where

$$M_{avg.} = (M_i + M_{i-1})/2$$

and c is the distance from the centroidal axis to the extreme fiber.

Recalling that $M_i = F\ell \mathcal{M}_i$, we may say that

$$\epsilon_{bi} = (F\ell c/2EI)(\mathcal{M}_i + \mathcal{M}_{i-1})$$

which reduces to

$$\epsilon_{bi} = (c/2n^2\ell)(\mathcal{M}_i + \mathcal{M}_{i-1})$$

However, $\ell = r\rho$, so that

$$\epsilon_{bi} = (c/2n^2 r\rho)(\mathcal{M}_i + \mathcal{M}_{i-1})$$

It is also necessary to make a choice, at this point, of the value to be used for c/r. We know that for a thin walled, hollow cylindrical cross section, $c/r = (2)^{1/2}$, while for a solid cylindrical cross section, $c/r = 2$. The value used in this investigation is $c/r = 1.5$.

Substituting this value in the equation above gives

$$\epsilon_{bi} = (0.75/n^2\rho)(\mathcal{M}_i + \mathcal{M}_{i-1})$$

from which

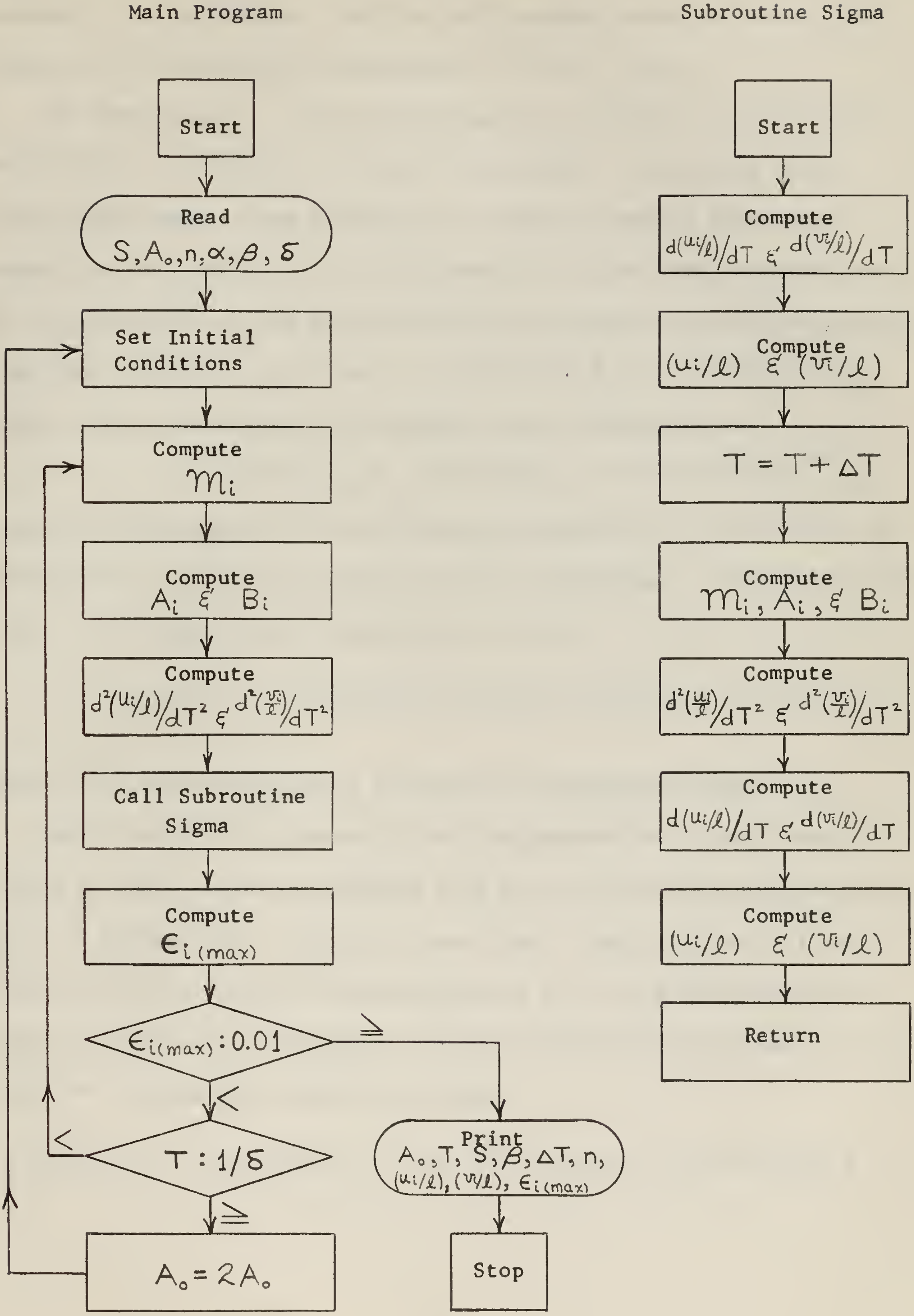
$$\epsilon_{i\,max} = \left| \Delta u_i/\ell + \theta_i^2/2 \right| + \left| \frac{0.75}{n^2\rho}(\mathcal{M}_i + \mathcal{M}_{i-1}) \right| \qquad (12)$$

3. The Numerical Solution

The equations of the system are solved by a numerical method of integration, utilizing Fortran programming and a Control Data Corporation 1604 high speed digital computer. A simplified block diagram of the basic program is shown in Fig. 5, and a complete program, together with the program notation, is given in Appendix III.

Referring now to Fig. 5, it will be seen that S, A_o, n, α, β,

Fig. 5. Block Diagram of Basic Computer Program


Main Program
Subroutine Sigma
Start
Read
$S, A_o, n, \alpha, \beta, \delta$
Set Initial Conditions
Compute
m_i
Compute
A_i & B_i
Compute
$d^2(u_i/\ell)/dT^2$ & $d^2(v_i/\ell)/dT^2$
Call Subroutine Sigma
Compute
$\epsilon_{i(max)}$
$\epsilon_{i(max)} : 0.01$
$\geq$
$<$
$T : 1/\delta$
$<$
$\geq$
$A_o = 2A_o$
Print
$A_o, T, S, \beta, \Delta T, n, (u_i/\ell), (v_i/\ell), \epsilon_{i(max)}$
Stop
Start
Compute
$d(u_i/\ell)/dT$ & $d(v_i/\ell)/dT$
Compute
(u_i/ℓ) & (v_i/ℓ)
$T = T + \Delta T$
Compute
$m_i, A_i,$ & B_i
Compute
$d^2(u_i/\ell)/dT^2$ & $d^2(v_i/\ell)/dT^2$
Compute
$d(u_i/\ell)/dT$ & $d(v_i/\ell)/dT$
Compute
(u_i/ℓ) & (v_i/ℓ)
Return

and δ are given as input data to the program. (δ is the ratio τ_1/T_t, where T_t is the arbitrarily selected dimensionless time at which the program is to terminate). For the basic program shown in Fig. 5, the value of A_0 is that which corresponds to $P_1/P_E = 1.0$.

At some time, T_1, when the values of $d^2(u_i/\ell)/dT^2$, $d^2(v_i/\ell)/dT^2$, $d(u_i/\ell)/dT$, $d(v_i/\ell)/dT$, (u_i/ℓ), and (v_i/ℓ) have been found, the program jumps to the subroutine in order to compute these same quantities for time $T_2 = T_1 + \Delta T$, where ΔT is the time increment used. As a first estimate, the accelerations are assumed to be constant during the time increment, ΔT; that is, $d^2(u_i/\ell)/dT^2\Big|_{T_2} = d^2(u_i/\ell)/dT^2\Big|_{T_1}$ Based on this assumption, the velocity may be calculated as

$$d(u_i/\ell)/dT\Big|_{T_2} = d(u_i/\ell)/dT\Big|_{T_1} + \Delta T \cdot d^2(u_i/\ell)/dT^2\Big|_{T_1}$$

(Only the development for the x-direction quantities is shown here; the development is precisely parallel for the y-direction) Having the velocity, we now compute the displacement at T_2 as

$$(u_i/\ell)\Big|_{T_2} = (u_i/\ell)\Big|_{T_1} + (\Delta T/2)\left[d(u_i/\ell)/dT\Big|_{T_1} + d(u_i/\ell)/dT\Big|_{T_2}\right]$$

which will be recognized as a trapezoidal integration method.

With the first estimates of the displacements at T_2 now known, the forces A_i and B_i may be calculated from eqs. (7) and (8) (with the exception, of course, of A_1, which is specified). Eqs. (10) and (11) are then utilized to find the accelerations at T_2. With these values in hand, the program now recomputes the velocities and displacements at T_2, using the trapezoidal integration scheme:

$$d(u_i/\ell)/dT\Big|_{T_2} = d(u_i/\ell)/dT\Big|_{T_1} + (\Delta T/2)\left[d^2(u_i/\ell)/dT^2\Big|_{T_1} + d^2(u_i/\ell)/dT^2\Big|_{T_2}\right]$$

and

$$(u_i/\ell)\big|_{T_2} = (u_i/\ell)\big|_{T_1} + (\Delta T/2)\left[d(u_i/\ell)/dT\big|_{T_1} + d(u_i/\ell)/dT\big|_{T_2}\right]$$

At this point, there is a return to the main program, and the $\epsilon_{i(max)}$ are calculated from eq. (12). These $\epsilon_{i(max)}$ values are then compared to the failure strain, 0.01. If any of the $\epsilon_{i(max)}$ is equal to or greater than 0.01, a printout of the quantities shown (in Fig. 5) is made and the program stops. If none is equal to or greater than the failure strain, the program compares the elapsed time with the specified terminate time. If the elapsed time is equal to or greater than the terminate time, the amplitude of the applied load is doubled, the initial conditions are reset, and the program commences again with the new value of applied load. If the elapsed time has not yet reached the terminate time, the computation is continued with the original value of A_0 .

In this manner, a rough set of failure values of P_1 / P_E is found. In order to refine these, a program such as the one given in Appendix III is used. It is basically the same as the one diagrammed in Fig. 5, with the following exceptions:

(a) the input value of A_0 is that which was found to give failure using the basic program

(b) the program does not stop after the printout of failure values. Instead, the A_0 value is decreased by a certain amount and a jump is made back to the point at which the initial conditions are set. The run continues until a non-failure value of A_0 is found.

(c) the program stops after the time comparison, when $T \geq 1/\delta$, instead of continuing with a doubled value of A_0 .

Using this second program, any desired degree of refinement in the failure values of P_1/P_E may be attained.

The program shows a marked sensitivity to the time increment, ΔT, used in the integration scheme. For $\beta \geq 5$, the time increment has been computed from

$$\Delta T = \frac{\pi/2}{n^2 \rho (0.4\beta)}$$

which gives an equal number of time intervals during the force pulse, for any $\beta \geq 5$. For $\beta < 5$, it was found that

$$\Delta T = \frac{\pi/2}{2 n^2 \rho}$$

which corresponds in real time to $\Delta t = \frac{1}{2}(\ell/s)$, gives good accuracy (as judged from a work - energy comparison which is described in the following section).

The computer program also shows some sensitivity to the number of length increments n used in the model. An investigation of this problem for values of n between 5 and 40 reveals that n = 20 gives good results without requiring excessive computer time.

---

4. Discussion of Results

Solutions for the failure value of P_1/P_E have been obtained for three values of S (column slenderness ratio), and for a range of β from 0.1 to 80. These results are summarized in Table 1 and are shown graphically in Fig. 6.

It should be noted that the values of P_1/P_E given in the table and plotted in Fig. 6 are values midway between the minimum P_1/P_E's giving failure and the maximum non-failure values. Adding the tolerance, Φ, to the tabulated P_1/P_E values gives the minimum failure P_1/P_E's found, while subtracting Φ gives the maximum non-failure values.

TABLE 1

Summarized Results

$\beta \downarrow$	S = 50			S = 100			S = 150		
	$\frac{P_1}{P_E}$	Φ	$\frac{\epsilon_c}{0.01}$	$\frac{P_1}{P_E}$	Φ	$\frac{\epsilon_c}{0.01}$	$\frac{P_1}{P_E}$	Φ	$\frac{\epsilon_c}{0.01}$
0.1	0.925	0.005	0.36	1.01	0.01	0.135	1.03	0.01	0.055
0.2	0.925	0.005	0.35	1.04	0.01	0.08	1.11	0.01	0.08
0.5	1.01	0.01	0.32	1.31	0.01	0.08	1.45	0.01	0.08
1	1.43	0.01	0.31	2.175	0.025	0.095	2.625	0.025	0.08
2	1.97	0.01	0.93	5.05	0.05	0.08	6.50	0.10	0.02
5	1.43	0.01	0.94	7.225	0.025	0.89	11.30	0.10	0.39
7	-	-	-	7.45	0.5	0.94	13.875	0.125	0.48
10	1.13	0.01	0.91	5.725	0.025	0.93	16.625	0.125	0.81
15	-	-	-	4.35	0.5	0.64	12.875	0.125	0.93
20	1.13	0.01	0.86	3.45	0.5	0.65	11.10	0.10	0.93
30	-	-	-	3.75	0.5	0.66	8.35	0.05	0.74
40	1.13	0.01	0.94	4.05	0.5	0.78	8.25	0.05	0.70
80	1.15	0.01	0.95	4.55	0.5	0.95	9.70	0.10	0.93

NOTE: P_1/P_E values listed are average of the min. failure value and the max. non-failure value at the particular β. Φ is the tolerance in the P_1/P_E values.

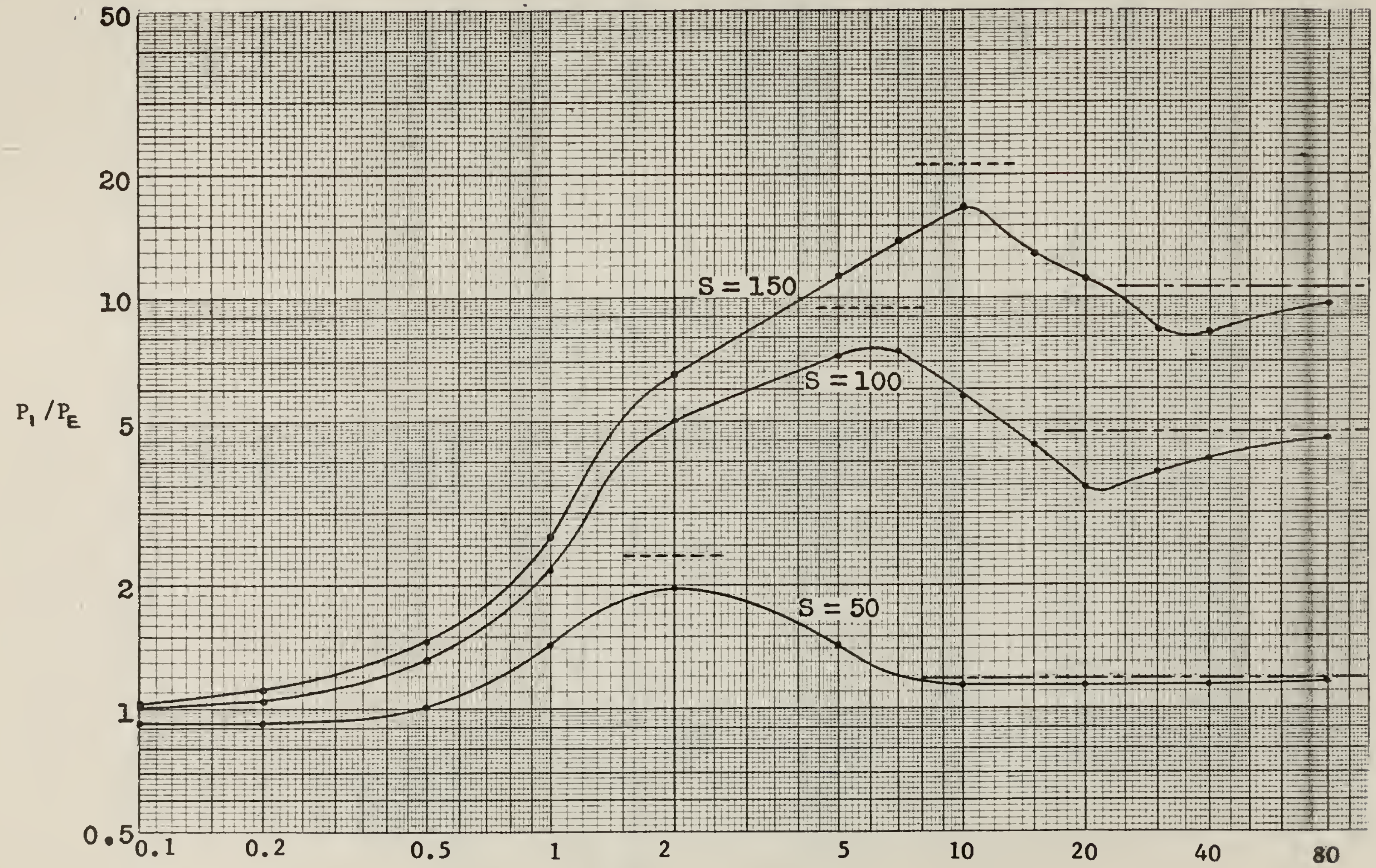


Fig. 6. Failure values of P_1/P_E vs. β for column slenderness ratios of 50, 100, and 150. Failure based on extreme fiber strain = 0.01. Dotted lines (- - - -) indicate max. P_1/P_E values (static loading with no bending). Dotted and dashed lines (—— - —— - ——) indicate max. P_1/P_E values for $\beta \geq S/2\pi$.

It is possible to make some predictions regarding the values of P_1/P_E which should obtain at both the high and low ends of the β range, and also, to predict the maximum values of P_1/P_E. First consider the case for $\beta = 0.1$, which is approaching the static loading case. If the failure criterion of $\epsilon_{max} = 0.01$ is applied to the secant formula for eccentrically loaded columns, the following is obtained:

$$\sigma_{max} = \frac{P_1}{A}\left[1 + \frac{ec}{r^2}\sec\left(S\left\{\frac{P_1}{4EA}\right\}^{1/2}\right)\right]$$

which becomes, on multiplication of both sides by A/P_E

$$\frac{\sigma_{max} A}{P_E} = \frac{P_1}{P_E}\left[1 + \frac{ec}{r^2}\sec\left(S\left\{\frac{P_1}{4EA}\right\}^{1/2}\right)\right]$$

However,

$$\frac{\sigma_{max} A}{P_E} = \epsilon_{max}\left(\frac{S}{\pi}\right)^2 \quad \text{and} \quad EA = P_E\left(\frac{S}{\pi}\right)^2$$

so that

$$\epsilon_{max}\left(\frac{S}{\pi}\right)^2 = \frac{P_1}{P_E}\left[1 + \frac{ec}{r^2}\sec\left(S\left\{\frac{P_1\pi^2}{4P_E S^2}\right\}^{1/2}\right)\right]$$

Upon substitution of the selected values of $e/r = 0.05$ and $c/r = 1.5$, this becomes

$$\epsilon_{max}\left(\frac{S}{\pi}\right)^2 = \frac{P_1}{P_E}\left[1 + 0.075\sec\left(\frac{\pi}{2}\left\{\frac{P_1}{P_E}\right\}^{1/2}\right)\right]$$

This equation, when solved by a trial and error method, yields the following values of P_1/P_E, compared here with the computer solution values for $\beta = 0.1$, for the three slenderness ratios:

Slenderness ratio S	Secant Formula Value P_1/P_E	Computer Solution Value P_1/P_E
50	0.943	0.925
100	0.989	1.01
150	0.996	1.03

Another value of P_1/P_E which may be predicted analytically is the

maximum load which the column can support without lateral deflection. If the column is loaded rapidly enough so that the amount of lateral deflection at failure is insignificant, then the column may be treated as an eccentrically loaded compression member, and the limiting load may be computed from

$$\epsilon_{max} = \frac{\sigma_{max}}{E\left(1+\frac{ec}{r^2}\right)} = \frac{(P_1/P_E)_{max}}{\frac{AE}{P_E}\left(1+\frac{ec}{r^2}\right)}$$

which gives

$$(P_1/P_E)_{max} = \epsilon_{max}\left(\frac{S}{\pi}\right)^2\left(\frac{1}{1+\frac{ec}{r^2}}\right)$$

The solution of this equation for S = 50, 100, and 150, $\epsilon_{max} = 0.01$, and $ec/r^2 = 0.075$ is compared below with the maximum values of P_1/P_E taken from the curves of Fig. 6.

Slenderness ratio S	Upper Limit (P_1/P_E) max	Computer (P_1/P_E) max
50	2.36	1.97
100	9.42	7.55
150	21.20	16.625

In each case, the computed value falls below the limiting value. This is due to the fact that bending action is allowed in the computer solution, whereas the theoretical solution assumes that only axial compression takes place, and also, to the fact that the computer solution is based on dynamic, rather than static loading.

The limiting value of P_1/P_E which should be found at the high end of the β range may be predicted from a consideration of the travel and reflection of an elastic compression wave in the column. For a wave having the shape of a half sine pulse and a duration of $\tau_f/2$, the following will be noted:

(a) the wave front will travel from the loaded end of the the column to the pinned end in a time L/s.

(b) at the pinned end, the original pulse will be reflected as a compression pulse traveling back toward the loaded end. This pulse will add to the original.

(c) the compression pulse thus reflected will travel back to the loaded end of the column where it will again be reflected, this time as a tension pulse which subtracts from the sum of the original and the first reflection.

From this analysis it may be seen that if the loading is sufficiently rapid that the first tension pulse reflected from the loaded end does not arrive at the pinned end before the peak value of the original compressive pulse, then the maximum axial force at the pinned end will be twice the peak value of the applied load.

Thus it is readily seen that, for $\tau_f/4 \leq 2L/s$ (or $\beta = \tau_1 / \frac{\tau_f}{2} \geq S/2\pi$), the maximum value of P_1/P_E which the column can support will be one-half the (P_1/P_E)max value previously predicted. The applicable values of β and $\frac{1}{2}(P_1/P_E)_{max}$ are given below for the three slenderness ratios.

S	β	$\frac{1}{2}(P_1/P_E)$max
50	7.95	1.18
100	15.90	4.71
150	23.85	10.60

These limiting values of P_1/P_E are shown on Fig. 6 as the dotted and dashed (—— - ——— - ——— -) lines, with the left hand ends of the lines indicating the minimum values of β for which these values hold. It will be noted that the computer solution values for all three slenderness ratios are below these limiting values, but appear to approach the limits asymptotically as β increases. Once again this is due to the

bending which has been allowed in the computer solution, but which is not considered in the limit analysis.

The excellent agreement shown between the computer and theoretical solutions for the three cases discussed above has been further enhanced by three additional checks which have been made on the adequacy of the model and computer program to give reliable results.

In Appendix I, the computer solution for a constant load is compared with the theoretical solution, given by the secant formula, for the same load. The computer solution gives deflections which oscillate about some average deflection curve, due to the fact that no damping is included in the model. The "static load" deflections for the computer solution have been calculated as the average of the maximum and minimum values. It will be seen that the deflection curves compare quite favorably, even though the computer solution is made with only ten length increments in the model, instead of the twenty which are used for all failure predictions. Also, the dynamic nature of the model response tends to give larger deflection values than those predicted by the secant formula.

Appendix II contains a graphical comparison of the theoretical and computer solution for the travel of an elastic strain wave down the column. It will be noted that the agreement is very good, in spite of the fact that the model used in the computer solution was eccentrically loaded and was allowed to bend. The amount of lateral deflection is extremely small, however, and does not affect the validity of this check.

A continuous check, comparing the work done on the column by the force pulse with the total energy stored in column as strain (potential) energy and kinetic energy, is made during all computer solutions for failure values of P_1/P_E. These agree within an average value of less

than two per cent. The maximum discrepancy found was 6.62 per cent.

From the data given in Table 1 and the curves of Fig. 6, it appears that there are several regions in the failure curves which may be distinguished from each other on the basis of the type of failure; that is, whether the failure is primarily due to bending strain or to axial strain. The type of failure may best be judged from a consideration of the data giving centroidal axis strain, ϵ_c, as a fraction of the failure strain, 0.01. In the case of all three slenderness ratios, beginning with the minimum β value, there is an initial region of bending failure, followed by a transition region leading, in each case, to the maximum P_1/P_E values and a region of axial strain failures. This is followed by another transition region which leads to the final region of axial strain failures.

Fig. 7 illustrates the lateral deflection change which occurs during one of the transitions from bending to axial strain failure. In this figure, the lateral deflections at the time of failure have been plotted for $\beta = 2$ ($\epsilon_c = 2\%$ of 0.01) and $\beta = 7$ ($\epsilon_c = 48\%$ of 0.01) for $S = 150$. It will be noted that the maximum lateral deflection for $\beta = 2$ is more than 14 times as large as the maximum for $\beta = 7$.

From the data compiled in Appendix V, it is possible to evaluate the approximations and assumptions made in developing the equations for the system. If this is done, the following is found:

(a) the maximum error in $\sin \theta_i = \theta_i$ is 1.61 per cent

(b) the maximum error in $\cos \theta_i = 1 - \theta_i^2/2$ is less than one per cent

(c) the maximum error in $1 - \Delta l_i/l = 1$ is less than one per cent

(d) the maximum error in $\Delta v_i/l = \theta_i$ is 1.56 per cent

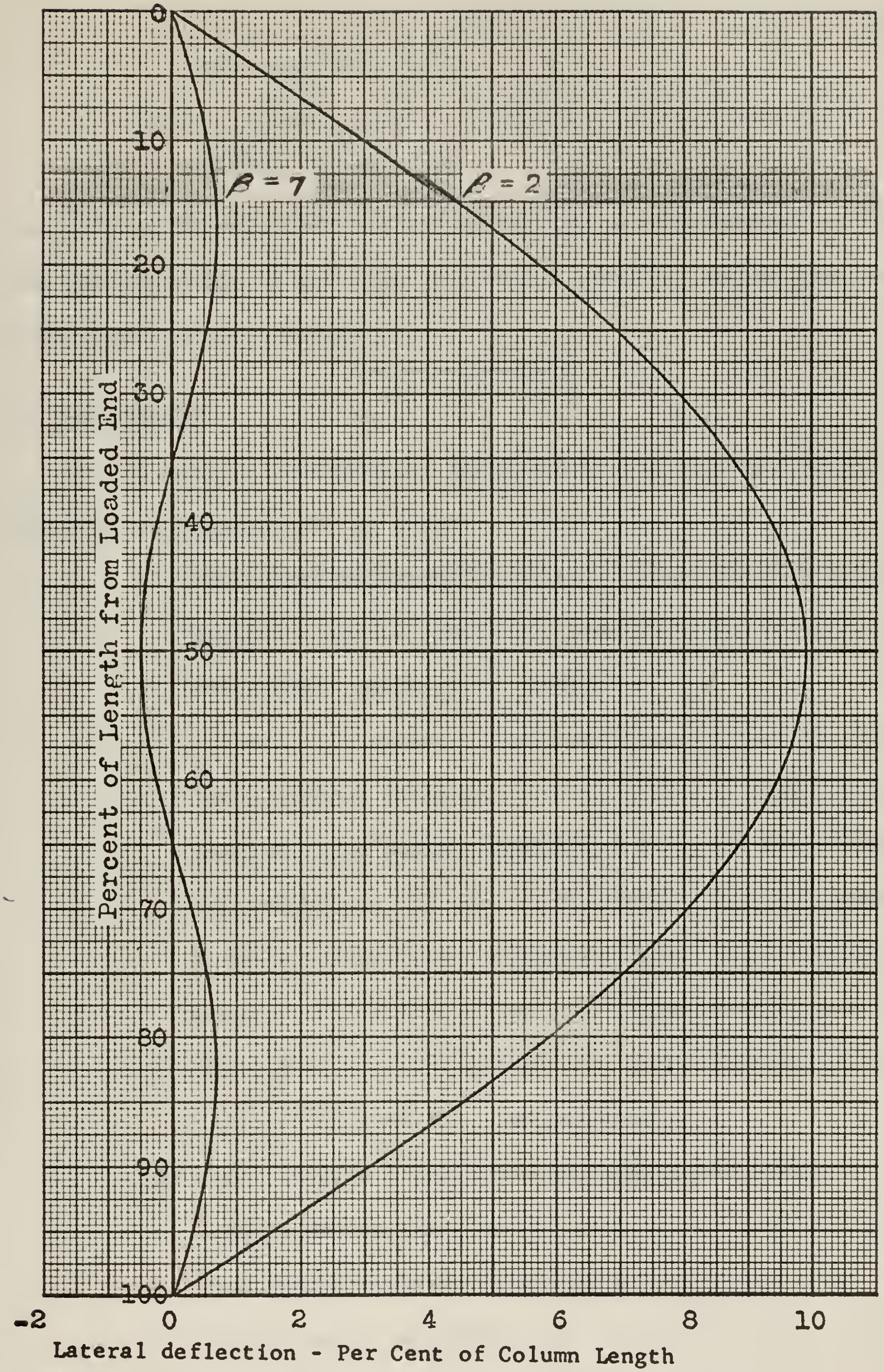


Fig. 7. Lateral deflections, at failure, of a column of slenderness ratio 150, for $\beta = 2$ ($P_1/P_E = 6.60$) and $\beta = 7$ ($P_1/P_E = 14.00$).

(e) $\theta_i^2/2$ has a maximum value of 0.05, which is small compared to 1.00

(f) the maximum value of $|\Delta u_i/\ell|$ is 0.05, which is small in comparison with 1.00

(g) the product $(\theta_i^2/2)\cdot(\Delta u_i/\ell)$ is less than 3% of the sum $(\theta_i^2/2)+(\Delta u_i/\ell)$

From this is appears that the assumptions and approximations which have been made in deriving the equations are reasonable, and do not produce gross errors in the results.

5. Conclusions

On the basis of the results discussed in the previous section, the following conclusions may be drawn:

(a) The lumped parameter model in combination with a high speed digital computer provides a powerful tool for the study of the dynamic behavior of columns.

(b) Columns will support loads much greater than the Euler load, without failing, when subjected to rapid dynamic loading. The maximum load which a column will support is dependent upon the type of loading and the slenderness ratio, as well as the yield strength of the column material. The ability of a column to support large dynamic loads without failing is due to the delay in the development of lateral deflections caused by lateral inertia.

(c) Axial inertia effects become significant with this type of loading during the first transition from bending failure to axial strain failure, and remain important as the rapidity of loading is increased.

(d) The mode of column failure - that is, whether the failure is caused primarily by strain due to bending or to axial compression - varies in a distinct manner as the rapidity of loading is increased.

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APPENDIX I

Comparison of theoretical and computer solution deflections for static load. $S = 100$; $e/r = 0.05$; $P_1/P_E = 0.85$

The theoretical deflection curve for an eccentrically loaded column may be derived from the secant formula as follows:

$$\frac{v_{max}}{L} = \frac{e}{L}\left[\sec\left(\frac{P_1 L^2}{4EI}\right)^{1/2} - 1\right]$$

where v_{max} is the maximum deflection from the original position. Now, $P_1 = 0.85P_E = 0.85\,\pi^2 EI/L^2$, so that

$$\frac{v_{max}}{L} = \frac{e}{L}\left[\sec\left(\frac{0.85\,\pi^2}{4}\right)^{1/2} - 1\right]$$

However, $e/L = e/(rS)$, and $e/r = 0.05$, $S = 100$.

Therefore

$$\frac{v_{max}}{L} = \frac{0.05}{100}\left[7.2055\right]$$

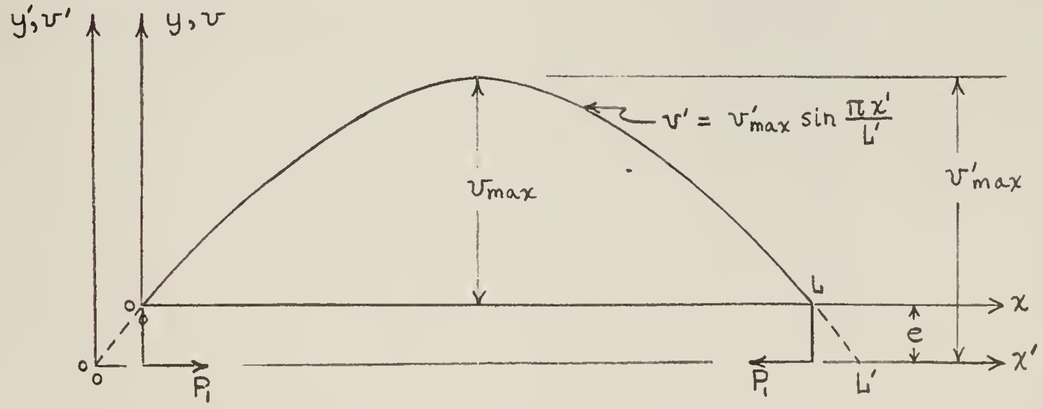
In order to get v_{max}/L on a percentage basis, multiply both sides by 100, giving

$$\frac{v_{max}}{L}(\%) = 0.05\left[7.2055\right] = 0.360275$$

The remainder of the theoretical deflection curve may now be calculated from

$$v/L(\%) = 100(v'_{max}/L)\sin(\pi x'/L') - 100\,(e/L)$$

which is derived from a consideration of the sketch below.

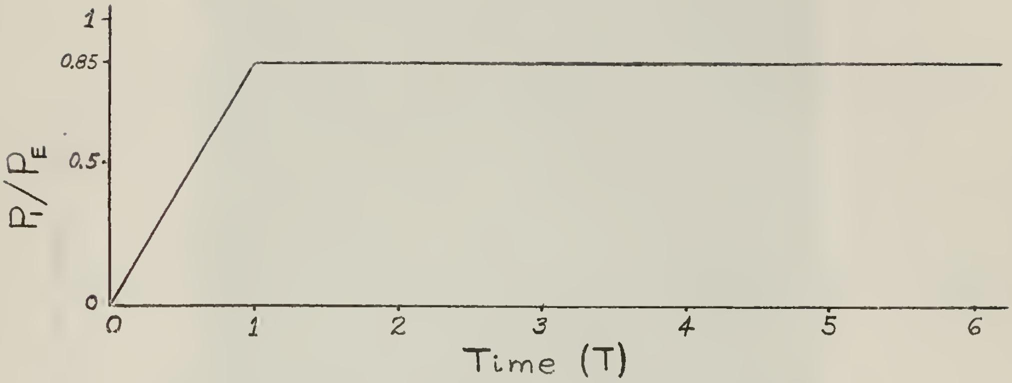


The theoretical deflections computed from this equation and the computer solution deflections are tabulated in the accompanying table (Table 2), and are compared graphically in Fig. 8. The computer solution does not yield a constant set of deflections, but rather, gives values which oscillate about some average deflection curve. This is due to the fact that no damping is included in the mathematical model. The static load deflections have been computed as the average of the maximum and minimum values given by the computer solution.

Table 2

Static Load Comparison

$S = 100;\ e/r = 0.05;\ P_1/P_E = 0.85$



Computer Solution Force-Time History

Per Cent Length (x/L x 100)	Theoretical Lateral Defl. (% L)	Maximum Computed Lateral Defl. (% L)	Minimum Computed Lateral Defl. (% L)	Average Computed Lateral Defl. (% L)
0	0	0	0	0
10	0.112	0.209	0.031	0.120
20	0.213	0.396	0.057	0.226
30	0.292	0.540	0.076	0.308
40	0.342	0.635	0.086	0.361
50	0.360	0.670	0.092	0.381
60	0.342	0.635	0.086	0.361
70	0.292	0.540	0.076	0.308
80	0.213	0.396	0.057	0.226
90	0.112	0.209	0.031	0.120
100	0	0	0	0

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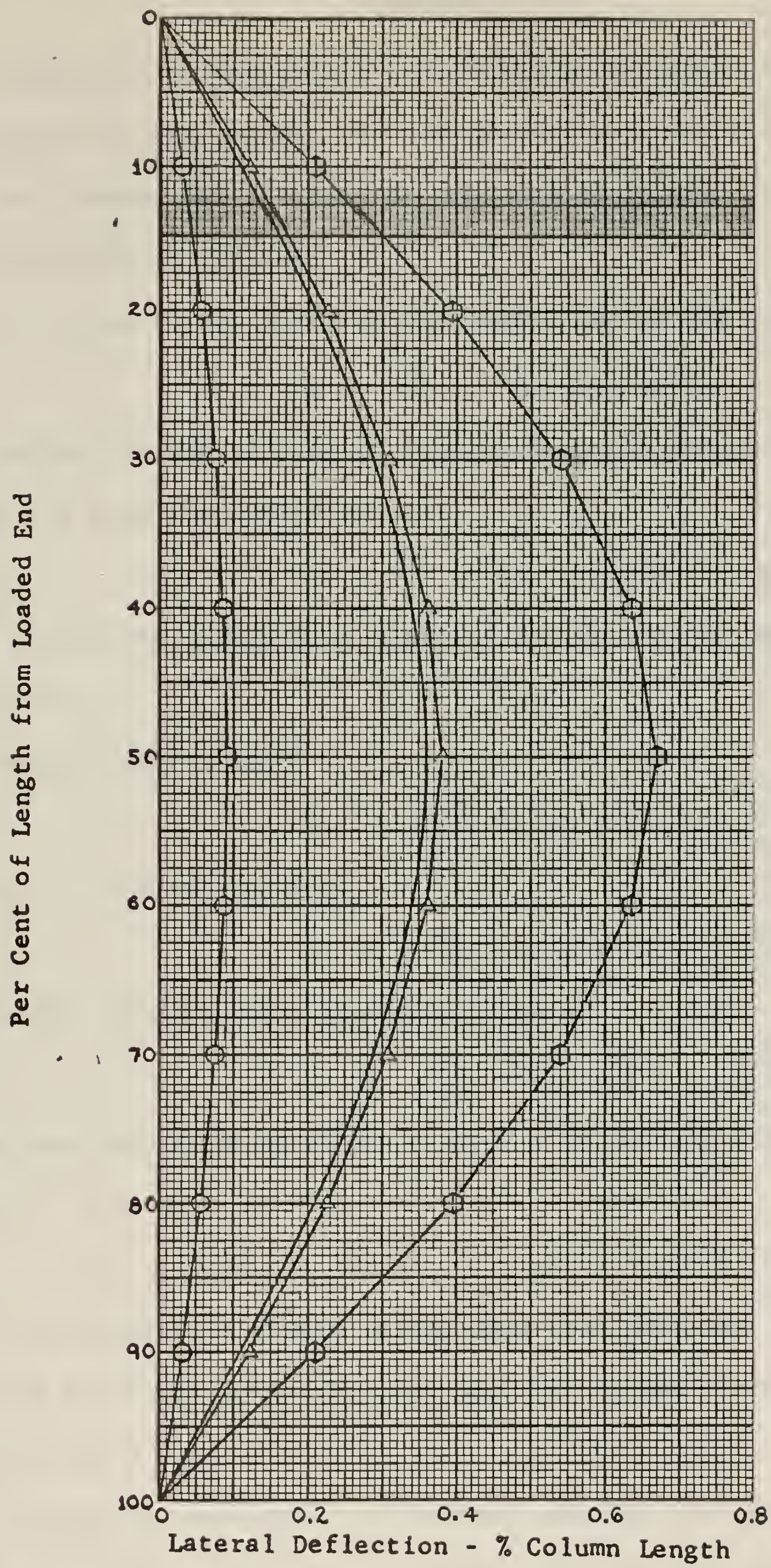

Fig. 8. Theoretical and Computer Solutions for lateral deflections of column with constant load. S = 100; e/r = 0.05. Solid line (———) is theoretical; **O** - minimum defl. computed; ⊕ - max. defl. computed; △ -average defl. computed.

APPENDIX II

Comparison of theoretical and computer solutions for an elastic strain wave in the column. $S = 50$, $\beta = 4$, $A_o = 10$, $T = 0.0392699$.

The theoretical solution for the strain at any point in the column may be derived as follows:

At a time T, the force at the loaded end of the column is

$$A_1 = \frac{P_1}{F} = A_o \sin(\pi\beta T)$$

and, neglecting the effects of lateral response motion, the force at some distance x from the loaded end is

$$A_i = \frac{P_i}{F} = A_o \sin\left[\pi\beta\left(T - \frac{x}{s\tau_1}\right)\right]$$

where x/s is the (real) time required for an elastic wave to travel a distance x.

Now, since $F = EA/S^2$,

$$P_i = \left(\frac{EA}{S^2}\right) A_o \sin\left[\pi\beta\left(T - \frac{x}{s\tau_1}\right)\right]$$

However, $\epsilon = P/EA$, so that

$$\epsilon_i = \frac{A_o}{S^2} \sin\left[\pi\beta\left(T - \frac{x}{s\tau_1}\right)\right]$$

Since $\tau_1 = 2LS/\pi s$, this may be written as

$$\epsilon_i = \frac{A_o}{S^2} \sin\left[\pi\beta\left(T - \frac{\pi x}{2LS}\right)\right]$$

which, upon substitution of $A = 10$, $S = 50$, $\beta = 4$, and $T = 0.0392699$, becomes

$$\epsilon_i = 4 \times 10^{-3} \sin\left[4\pi\left(0.0392699 - \frac{\pi x}{100L}\right)\right]$$

The theoretical strain values have been computed from this equation and are compared graphically with the values from the computer solution in Fig. 9.

It will be recognized in the last equation for ϵ_i that x must reach a value of 1.25L before $\epsilon_i = 0$. This means that the pulse has reached the

pinned end of the column and has been reflected back a distance of 0.25L. Thus, the values of ϵ_i computed for $1 \leq x/L \leq 1.25$ must be added to the values calculated for $1 \geq x/L \geq 0.75$ to give the resultant strain in the region $0.75 \leq x/L \leq 1$.

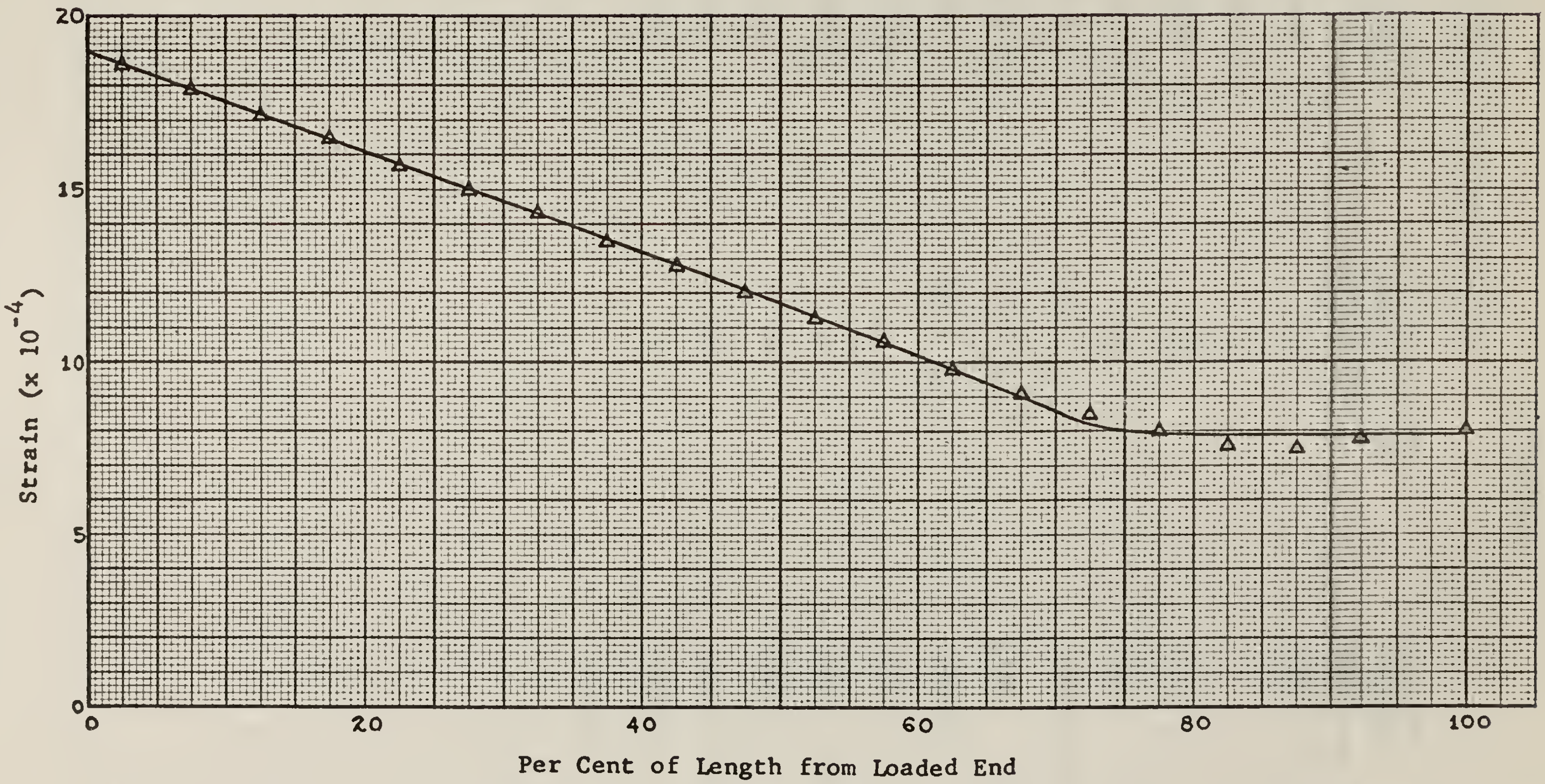

Fig. 9. Theoretical and computer solutions for elastic strain wave in column. S = 50, $\beta = 4$, $A_1 = 10 \sin (\pi \beta T)$, T = 0.0392699 = 50 ΔT. Solid line (———) is theoretical strain; Δ's are computed strains

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APPENDIX III

Computer program notation

Computer Program Term	Corresponding symbol in the basic eqs.
ALPHA (I)	A_i
ANGL (I)	θ_i
BENDM (I)	m_i
BETA (I)	B_i
DDX (I)	$d^2(u_i/\ell)/dT^2$
DDY (I)	$d^2(v_i/\ell)/dT^2$
ECC	$\frac{e}{\ell} = \frac{e}{r\rho}$
EXTSTR (I)	$\epsilon_{i\,(max)}$
FORCE	A_0
FAC1	α
FAC2	β
FAC3	δ

Computer program term	Corresponding symbol in the basic eqs.
INTV	n
J	$n + 1$
RHO	ρ
SLRT	S
STRAIN (I)	ϵ_{ci}
TIME	T
TIMEINT	ΔT
XACCL (I)	$d^2(u_i/\ell)/dT^2$
XDEFL (I)	(u_i/ℓ)
XVEL (I)	$d(u_i/\ell)/dT$
YACCL (I)	$d^2(v_i/\ell)/dT^2$
YDEFL (I)	(v_i/ℓ)
YVEL (I)	$d(v_i/\ell)/dT$

```
      PROGRAM COL/
PROGRAM TO DETERMINE COLUMN BEHAVIOR WHEN HALF SINE PULSE LOAD IS
APPLIED WITH SMALL ECCENTRICITY TO INITIALLY STRAIGHT COLUMN
     0DIMENSION ALPHA(60),BETA(60), XACCL(60),YACCL(60), XVEL(60),
     1YVEL(60), XDEFL(60),YDEFL(60),BENDM(60),ANGL(60),STRAIN(60),
     2 U(1), P(1), EXISTR(60)
     0 COMMON INTV,J,TIMEINT,XACCL,XVEL,XDEFL,YACCL,YVEL,YDEFL,
     1 AINTV,RHO,ANGL,BENDM,ALPHA,BETA,TIME,ECC,FAC2,FORCE,C1,C2,C3,
     2 WORK
      INTV = 20
      READ 100, SLRT, FAC1, FAC2, FAC3, FORCE, ECC
      READ 110, A
100   FORMAT (5E9.2, E20.6)
110   FORMAT (E9.2)
      J = INTV + 1
      AINTV = INTV
      RHO = SLRT/AINTV
190   DO 200 I = 1,J
      ANGL(I) = 0.
      XVEL(I) = 0.
      YVEL(I) = 0.
      XDEFL(I) =0.
      YDEFL(I) = 0.
200   BENDM(I) = 0.
       FORCE = FORCE - A * 9.87
      WORK = 0.
      TIME = 0.
      ALPHA(1) = 0.
      FAC1 = 0.40 * FAC2
      TIMEINT = (1.570796)/((AINTV**2)*RHO*FAC1)
      C1 = (0.405285)*(AINTV**2)
      C2 = 2. * C1
      C3 = TIMEINT/2.
210   DO 220 I = 2,INTV
220   BENDM(I) = (AINTV**2)*(ANGL(I) -ANGL(I+1))
      DO 230 I = 2,J
2300ALPHA(I) = -(AINTV**2)*(RHO**2)*(XDEFL(I) -XDEFL(I-1)
```

```
     1 +(ANGL(I)**2)/2.)
      BENDM(J) = ALPHA(J) * ECC
      DO 235 I = 2,J
235   BETA(I) = ANGL(I) * ALPHA(I) - (BENDM(I) -BENDM(I-1))
      BETA(1) = BETA(2)
      ALPHA(J+1) = ALPHA(J)
      BETA(J+1) = BETA(J)
      DO 240 I = 2,J
      XACCL(I) = C1*(ALPHA(I) - ALPHA(I+1))
240   YACCL(I) = C1 * (BETA(I) -BETA(I+1))
      XACCL(1) = C2 * (ALPHA(1) -ALPHA(2))
      YACCL(1) = 0.
      CALL SIGMA
250   DO 255 I = 2,J
255   STRAIN(I) = -(XDEFL(I) -XDEFL(I-1) +(ANGL(I)**2)/2.)
      STRAIN(1) = 0.
       DO 256 I = 2,INTV
256   BENDM(I) = (AINTV**2)*(ANGL(I)-ANGL(I+1))
      DO 260 I = 2,J
2600EXTSTR(I)= ABSF ( STRAIN(I)) + ( 0.75/(RHO*AINTV**2))
     1 * ABSF (BENDM(I) + BENDM(I-1))
      EXTSTR(1) = 0.
      I = 1
900   I = I + 1
      IF ( I - J ) 905,905,950
950   IF( TIME - 1./FAC3 ) 210,710,710
905   IF ( EXTSTR(I) - 0.01 ) 900,910,910
910   SUMKEN = 0.
      VEN = 0.
      BENEN = 0.
      COMEN = 0.
      TOTEN = 0.
      DO 915 I = 2,INTV
      VEN=VEN+(1.233701/AINTV**2)*(XVEL(I)**2+YVEL(I)**2)
915   BENEN=(0.50/AINTV**2)*(BENDM(I)**2) + BENEN
      SUMKEN = VEN + (0.616850/AINTV**2)*(XVEL(1)**2)
      DO 916 I = 2,J
```

```
916 COMEN=COMEN+(0.50)*(AINTV**2)*(RHO**2)*(STRAIN(1)**2)
    TOTEN = SUMKEN + BENEN + COMEN
    PRINT 300
3000FORMAT (6H  SLRT7X,5HFORCE7X,3HECC7X,4HFAC26X,7HTIMEINT3X,
   14HINTV//)
    PRINT 400 (SLRT, FORCE, ECC, FAC2, TIMEINT, INTV)
400 FORMAT ( 1PE9.2, 1P4E11.2, I7////)
    PRINT 500
500 FORMAT (20H                   TIME//)
     PRINT 510, (TIME)
510 FORMAT (1PE20.7, 1H //)
    PRINT 550
550 FORMAT (5H    I8X,5HXDEFL8X,5HYDEFL7X,6HEXTSTR//)
    PRINT 600, ( I, XDEFL(I), YDEFL(I), EXTSTR(I), I = 1,J)
600 FORMAT (I5, 1P3E13.2)
    PRINT 601
601 FORMAT ( 1H0 )
    PRINT 605
605 FORMAT (20H                  TOTEN16X,4HWORK//)
    PRINT 606 ( TOTEN, WORK )
606 FORMAT ( 1P2E20.7 )
    PRINT 800
800 FORMAT (1H1)
    GO TO 190
710 STOP
    END
     SUBROUTINE SIGMA
   0DIMENSION ALPHA(60),BETA(60), XACCL(60),YACCL(60), XVEL(60),
   1YVEL(60), XDEFL(60),YDEFL(60),BENDM(60),ANGL(60),STRAIN(60),
   2U(1),P(1),  DDX(60), DDY(60), XSPD(60), YSPD(60), X(60), Y(60)
   0 COMMON INTV,J,TIMEINT,XACCL,XVEL,XDEFL,YACCL,YVEL,YDEFL,
   1 AINTV,RHO,ANGL,BENDM,ALPHA,BETA,TIME,ECC,FAC2,FORCE,C1,C2,C3,
   2 WORK
    U(1) = XDEFL(1)
    P(1) = ALPHA(1)
    DO 10 I = 1,J
    XSPD(I) = XVEL(I) + TIMEINT * XACCL(I)
```

```
   YSPD(I) = YVEL(I) + TIMEINT * YACCL(I)
   X(I) = XDEFL(I)                 +C3 * (XVEL(I) + XSPD(I))
10 Y(I) = YDEFL(I) + C3 * (YVEL(I) + YSPD(I))
   DO 20 I = 2,J
20 ANGL(I) = Y(I) - Y(I - 1)
   TIME = TIME + TIMEINT
   IF (TIME * FAC2 -1.) 25,26,26
25 ALPHA(1) = FORCE * SINF(FAC2 * TIME * 3.141593)
   GO TO 27
26 ALPHA(1) = 0.
27 DO 28 I = 2,J
28 ALPHA(I) = -(AINTV**2)*(RHO**2)*(X(I)-X(I-1)+(ANGL(I)**2)/2.)
   ALPHA(J+1) = ALPHA(J)
   DO 30 I = 2,INTV
30 BENDM(I) = (AINTV**2)*(ANGL(I) -ANGL(I+1))
   BENDM(1) = ALPHA(1) * ECC
   BENDM(J) = ALPHA(J) * ECC
   DO 40 I = 2,J
40 BETA(I) = (ANGL(I)*ALPHA(I)) - (BENDM(I) - BENDM(I-1))
   BETA(1) = BETA(2)
   BETA(J+1) = BETA(J)
   DDX(1) =                    C2*(ALPHA(1) - ALPHA(2))
   DDY(1) = 0.
   DO 50 I= 2,J
   DDX(I) =                    C1*(ALPHA(I) -ALPHA(I+1))
50 DDY(I) =                    C1*(BETA(I) -BETA(I+1))
   DO 55 I= 1,J
   XSPD(I) = XVEL(I) +           C3*(XACCL(I) + DDX(I))
   YSPD(I) = YVEL(I) +           C3*(YACCL(I) + DDY(I))
   XDEFL(I) = XDEFL(I) +           C3*(XVEL(I) + XSPD(I))
   YDEFL(I) = YDEFL(I) +           C3*(YVEL(I) +YSPD(I))
   XVEL(I) = XSPD(I)
55 YVEL(I) = YSPD(I)
   DO 60 I = 2,J
60 ANGL(I) = YDEFL(I) - YDEFL(I-1)
   WORK = WORK + ( XDEFL(1) -U(1))*( P(1) + ALPHA(1))/2.
   RETURN
```

APPENDIX IV

Failure Data

Tabulated in the following pages are the failure data for the three slenderness ratios for all values of β which were investigated. These data are the x and y direction deflections and the extreme fiber strains at the time of failure, for the minimum failure values of P_1/P_E. The strain values given are the strains half-way between the point for which they are listed and the previous point. Thus, a strain listed opposite 50 per cent of the length is actually the strain at 47.5 per cent of the length.

Failure Deflection and Strain Data for Minimum Failure Value of P_1/P_E

$S = 50$

$T = 5.33$

$\beta = 0.1$

$\Delta T = 7.85 \times 10^{-4}$

$P_1/P_E = 0.930$

$n = 20$

Per cent of length from loaded end (%)	Longitudinal deflection (u_i/ℓ)	Lateral deflection (v_i/ℓ)	Extreme fiber strain
0	9.66×10^{-2}	0	0
5	9.06	0.694×10^{-1}	0.441×10^{-2}
10	8.46	1.37	0.537
15	7.90	2.00	0.628
20	7.36	2.59	0.714
25	6.86	3.11	0.791
30	6.40	3.55	0.858
35	5.97	3.91	0.914
40	5.57	4.17	0.956
45	5.19	4.33	0.985
50	4.83	4.38	1.00
55	4.46	4.33	1.00
60	4.08	4.17	0.985
65	3.69	3.91	0.956
70	3.26	3.55	0.913
75	2.80	3.11	0.857
80	2.30	2.59	0.790
85	1.76	2.00	0.713
90	1.19	1.37	0.627
95	0.605	0.694	0.536
100	0	0	0.440

Failure Deflection and Strain Data for Minimum Failure Value of P_1/P_E

$S = 50$

$T = 3.08$

$\beta = 0.2$

$\Delta T = 7.85 \times 10^{-4}$

$P_1/P_E = 0.930$

$n = 20$

Per cent of length from loaded end (%)	Longitudinal deflection (u_i/ℓ)	Lateral deflection (v_i/ℓ)	Extreme fiber strain
0	9.41×10^{-2}	0	0
5	8.81	0.716×10^{-1}	0.419×10^{-2}
10	8.23	1.41	0.518
15	7.67	2.07	0.614
20	7.14	2.67	0.703
25	6.65	3.21	0.783
30	6.21	3.67	0.853
35	5.79	4.03	0.911
40	5.41	4.30	0.955
45	5.06	4.47	0.985
50	4.71	4.52	1.00
55	4.37	4.47	1.00
60	4.01	4.30	0.985
65	3.63	4.03	0.956
70	3.22	3.67	0.912
75	2.77	3.21	0.854
80	2.28	2.67	0.785
85	1.75	2.07	0.705
90	1.19	1.41	0.616
95	0.601	0.716	0.521
100	0	0	0.422

Failure Deflection and Strain Data for Minimum Failure Value of P_1/P_E

$S = 50$ $T = 1.45$

$\beta = 0.5$ $\Delta T = 7.85 \times 10^{-4}$

$P_1/P_E \doteqdot 1.02$ $n = 20$

Per cent of length from loaded end (%)	Longitudinal deflection (u_l/ℓ)	Lateral deflection (v_l/ℓ)	Extreme fiber strain
0	9.00×10^{-2}	0	0
5	8.41	0.755×10^{-1}	0.385×10^{-2}
10	7.84	1.49	0.491
15	7.29	2.18	0.593
20	6.78	2.82	0.688
25	6.31	3.39	0.774
30	5.88	3.87	0.847
35	5.50	4.26	0.908
40	5.15	4.54	0.954
45	4.83	4.71	0.985
50	4.51	4.77	1.00
55	4.20	4.71	1.00
60	3.88	4.54	0.985
65	3.53	4.25	0.955
70	3.14	3.87	0.909
75	2.72	3.39	0.849
80	2.24	2.82	0.776
85	1.73	2.18	0.691
90	1.18	1.49	0.597
95	0.597	0.755	0.495
100	0	0	0.389

Failure Deflection and Strain Data for Minimum Failure Value of P_1/P_E

$S = 50$

$T = 0.837$

$\beta = 1$

$\Delta T = 7.85 \times 10^{-4}$

$P_1/P_E \doteqdot 1.44$

$n = 20$

Per cent of length from loaded end (%)	Longitudinal deflection (u_i/ℓ)	Lateral deflection (v_i/ℓ)	Extreme fiber strain
0	8.91×10^{-2}	0	0
5	8.35	0.755×10^{-1}	0.355×10^{-2}
10	7.79	1.49	0.467
15	7.27	2.18	0.574
20	6.77	2.82	0.673
25	6.32	3.39	0.762
30	5.90	3.87	0.838
35	5.53	4.25	0.902
40	5.18	4.54	0.950
45	4.86	4.71	0.982
50	4.55	4.76	0.999
55	4.24	4.70	1.00
60	3.92	4.53	0.985
65	3.56	4.25	0.955
70	3.17	3.86	0.909
75	2.74	3.38	0.851
80	2.26	2.81	0.779
85	1.74	2.18	0.696
90	1.19	1.48	0.603
95	0.601	0.753	0.501
100	0	0	0.395

Failure Deflection and Strain Data for Minimum Failure

Value of P_1/P_E

$S = 50$ $T = 0.222$

$\beta = 2$ $\Delta T = 7.85 \times 10^{-4}$

$P_1/P_E = 1.98$ $n = 20$

Per cent of length from loaded end (%)	Longitudinal deflection (u_i/ℓ)	Lateral deflection (v_i/ℓ)	Extreme fiber strain
0	1.72×10^{-1}	0	0
5	1.64	0.502×10^{-2}	0.832×10^{-2}
10	1.56	0.909	0.838
15	1.48	1.23	0.843
20	1.40	1.47	0.847
25	1.32	1.64	0.850
30	1.24	1.75	0.852
35	1.15	1.81	0.853
40	1.07	1.84	0.854
45	9.86×10^{-2}	1.86	0.856
50	9.01	1.86	0.859
55	8.14	1.86	0.865
60	7.27	1.85	0.876
65	6.40	1.84	0.891
70	5.51	1.79	0.910
75	4.61	1.70	0.930
80	3.70	1.55	0.950
85	2.79	1.32	0.969
90	1.86	0.988	0.984
95	0.932	0.552	0.995
100	0	0	1.00

Failure Deflection and Strain Data for Minimum Failure Value of P_1/P_E

$S = 50$ $T = 0.0950$

$\beta = 5$ $\Delta T = 7.85 \times 10^{-4}$

$P_1/P_E \doteq 1.44$ $n = 20$

Per cent of length from loaded end (%)	Longitudinal deflection (u_i/ℓ)	Lateral deflection (v_i/ℓ)	Extreme fiber strain
0	1.55×10^{-1}	0	0
5	1.49	2.08×10^{-3}	0.620×10^{-2}
10	1.43	3.46	0.642
15	1.37	4.19	0.662
20	1.30	4.33	0.680
25	1.24	3.98	0.695
30	1.17	3.25	0.707
35	1.10	2.32	0.718
40	1.02	1.34	0.734
45	9.47×10^{-2}	4.91×10^{-4}	0.765
50	8.70	-6.71×10^{-5}	0.795
55	7.91	-2.21×10^{-4}	0.818
60	7.10	5.82×10^{-5}	0.834
65	6.27	7.16×10^{-4}	0.845
70	5.43	1.62×10^{-3}	0.853
75	4.57	2.60	0.866
80	3.68	3.40	0.901
85	2.78	3.78	0.935
90	1.86	3.48	0.965
95	0.935	2.28	0.988
100	0	0	1.00

Failure Deflection and Strain Data for Minimum Failure Value of P_1/P_E

$S = 50$ $T = 0.389$

$\beta = 10$ $\Delta T = 3.93 \times 10^{-4}$

$P_1/P_E \doteqdot 1.14$ $n = 20$

Per cent of length from loaded end (%)	Longitudinal deflection (u_i/ℓ)	Lateral deflection (v_i/ℓ)	Extreme fiber strain
0	-1.16×10^{-1}	0	0
5	-1.16	-1.34×10^{-3}	5.43×10^{-4}
10	-1.14	-2.40	0.157×10^{-2}
15	-1.12	-2.89	0.247
20	-1.10	-2.60	0.326
25	-1.06	-1.44	0.393
30	-1.02	6.21×10^{-4}	0.450
35	-9.76×10^{-2}	3.45×10^{-3}	0.496
40	-9.25	6.78	0.528
45	-8.68	1.01×10^{-2}	0.583
50	-8.07	1.29	0.671
55	-7.41	1.44	0.749
60	-6.70	1.43	0.807
65	-5.96	1.25	0.837
70	-5.18	9.30×10^{-3}	0.839
75	-4.38	5.32	0.820
80	-3.55	1.49	0.862
85	-2.70	-1.34	0.933
90	-1.82	-2.57	0.982
95	-0.915	-2.05	1.00
100	0	0	0.996

Failure Deflection and Strain Data for Minimum Failure Value of P_1/P_E

$S = 50$

$T = 0.496$

$\beta = 20$

$\Delta T = 1.96 \times 10^{-4}$

$P_1/P_E = 1.14$

$n = 20$

Per cent of length from loaded end (%)	Longitudinal deflection (u_i/ℓ)	Lateral deflection (v_i/ℓ)	Extreme fiber strain
0	-9.23×10^{-2}	0	0
5	-9.22	-1.87×10^{-3}	1.01×10^{-4}
10	-9.20	-3.82	2.49
15	-9.15	-5.75	6.08
20	-9.08	-7.22	0.118×10^{-2}
25	-8.98	-7.57	0.190
30	-8.82	-6.26	0.252
35	-8.63	-3.30	0.278
40	-8.36	0.634	0.287
45	-8.03	4.39	0.376
50	-7.61	6.87	0.518
55	-7.11	7.38	0.619
60	-6.53	5.78	0.697
65	-5.86	2.49	0.753
70	-5.11	-1.72	0.774
75	-4.32	-5.82	0.826
80	-3.49	-8.77	0.920
85	-2.64	-9.74	0.978
90	-1.78	-8.43	1.00
95	-0.892	-5.02	0.996
100	0	0	0.975

Failure Deflection and Strain Data for Minimum Failure Value of P_1/P_E

$S = 50$ $T = 0.170$

$\beta = 40$ $\Delta T = 9.82 \times 10^{-5}$

$P_1/P_E = 1.14$ $n = 20$

Per cent of length from loaded end (%)	Longitudinal deflection (u_i/ℓ)	Lateral deflection (v_i/ℓ)	Extreme fiber strain
0	4.52×10^{-2}	0	0
5	4.55	1.23×10^{-3}	3.93×10^{-4}
10	4.59	2.23	4.73
15	4.60	2.98	2.47
20	4.56	3.48	5.26
25	4.53	3.55	5.77
30	4.52	3.12	3.68
35	4.53	2.33	2.35
40	4.55	1.50	1.86
45	4.59	0.843	5.44
50	4.61	0.457	4.27
55	4.63	0.444	4.20
60	4.58	0.805	6.41
65	4.43	1.24	0.160×10^{-2}
70	4.15	1.47	0.286
75	3.76	1.41	0.411
80	3.25	1.24	0.504
85	2.61	1.17	0.653
90	1.84	1.21	0.775
95	0.943	0.980	0.923
100	0	0	1.00

Failure Deflection and Strain Data for Minimum Failure Value of P_1/P_E

$S = 50$ $T = 0.0378$

$\beta = 80$ $\Delta T = 4.91 \times 10^{-5}$

$P_1/P_E \fallingdotseq 1.16$ $n = 20$

Per cent of length from loaded end (%)	Longitudinal deflection (u_i/ℓ)	Lateral deflection (v_i/ℓ)	Extreme fiber strain
0	2.32×10^{-2}	0	0
5	2.32	1.63×10^{-4}	2.48×10^{-5}
10	2.32	3.15	2.36
15	2.32	4.45	2.07
20	2.32	5.34	5.18
25	2.32	5.69	8.00
30	2.32	5.24	7.90
35	2.32	3.97	7.80
40	2.33	1.95	1.68×10^{-4}
45	2.31	−0.460	1.84
50	2.31	−2.36	6.80×10^{-5}
55	2.35	−3.10	4.00×10^{-4}
60	2.31	−2.01	4.20
65	2.28	0.365	3.36
70	2.33	1.91	5.66
75	2.37	1.17	5.33
80	2.32	−0.595	5.49
85	2.12	−0.622	2.13×10^{-3}
90	1.68	1.86	4.43
95	0.947	4.00	7.53
100	0	0	1.00×10^{-2}

Failure Deflection and Strain Data for Minimum Failure

Value of P_1/P_E

$S = 100$ $T = 5.50$

$\beta = 0.1$ $\Delta T = 7.85 \times 10^{-4}$

$P_1/P_E \doteq 1.02$ $n = 20$

Per cent of length from loaded end (%)	Longitudinal deflection (u_i/ℓ)	Lateral deflection (v_i/ℓ)	Extreme fiber strain
0	1.97×10^{-1}	0	0
5	1.76	0.187	3.30×10^{-3}
10	1.60	0.370	2.78
15	1.43	0.543	5.98
20	1.30	0.703	5.02
25	1.18	0.845	7.94
30	1.11	0.967	6.71
35	1.04	1.06	9.27
40	1.01	1.14	8.51
45	9.87×10^{-2}	1.18	9.99
50	9.80	1.19	9.39
55	9.66	1.18	1.00×10^{-2}
60	9.49	1.14	9.31×10^{-3}
65	9.11	1.06	9.29
70	8.55	0.967	8.33
75	7.70	0.845	7.82
80	6.60	0.703	6.60
85	5.21	0.543	5.71
90	3.62	0.370	4.33
95	1.85	0.187	3.14
100	0	0	1.74

Failure Deflection and Strain Data for Minimum Failure Value of P_1/P_E

$S = 100$ $T = 3.47$

$\beta = 0.2$ $\Delta T = 3.93 \times 10^{-4}$

$P_1/P_E \doteqdot 1.05$ $n = 20$

Per cent of length from loaded end (%)	Longitudinal deflection (u_i/ℓ)	Lateral deflection (v_i/ℓ)	Extreme fiber strain
0	2.09×10^{-1}	0	0
5	1.89	0.195	1.63×10^{-3}
10	1.70	0.386	3.04
15	1.53	0.566	4.39
20	1.38	0.733	5.66
25	1.26	0.881	6.82
30	1.17	1.01	7.83
35	1.11	1.11	8.67
40	1.07	1.19	9.32
45	1.05	1.23	9.76
50	1.04	1.25	9.99
55	1.03	1.23	1.00×10^{-2}
60	1.02	1.19	9.78×10^{-3}
65	9.79×10^{-2}	1.11	9.35
70	9.19	1.01	8.70
75	8.29	0.883	7.86
80	7.10	0.734	6.86
85	5.62	0.567	5.70
90	3.90	0.386	4.42
95	2.00	0.196	3.06
100	0	0	1.65

Failure Deflection and Strain Data for Minimum Failure

Value of P_1/P_E

$S = 100$ $T = 1.58$

$\beta = 0.5$ $\Delta T = 3.93 \times 10^{-4}$

$P_1/P_E \doteq 1.32$ $n = 20$

Per cent of length from loaded end (%)	Longitudinal deflection (u_i/ℓ)	Lateral deflection (v_i/ℓ)	Extreme fiber strain
0	2.10×10^{-1}	0	0
5	1.90	0.197	1.57×10^{-3}
10	1.71	0.389	2.98
15	1.54	0.571	4.35
20	1.39	0.739	5.63
25	1.27	0.889	6.80
30	1.18	1.02	7.82
35	1.12	1.12	8.68
40	1.08	1.20	9.33
45	1.06	1.24	9.78
50	1.05	1.26	1.00×10^{-2}
55	1.04	1.24	1.00
60	1.02	1.20	9.77×10^{-3}
65	9.88×10^{-2}	1.12	9.32
70	9.27	1.02	8.67
75	8.37	0.889	7.83
80	7.17	0.740	6.82
85	5.68	0.571	5.66
90	3.94	0.389	4.37
95	2.02	0.197	3.00
100	0	0	1.57

Failure Deflection and Strain Data for Minimum Failure Value of P_1/P_E

$S = 100$ $T = 0.931$

$\beta = 1$ $\Delta T = 3.93 \times 10^{-4}$

$P_1/P_E \doteqdot 2.20$ $n = 20$

Per cent of length from loaded end (%)	Longitudinal deflection (u_i/ℓ)	Lateral deflection (v_i/ℓ)	Extreme fiber strain
0	2.16×10^{-1}	0	0
5	1.95	0.203	1.25×10^{-3}
10	1.75	0.400	2.73
15	1.57	0.588	4.15
20	1.42	0.761	5.49
25	1.29	0.915	6.70
30	1.20	1.05	7.76
35	1.14	1.15	8.63
40	1.11	1.23	9.31
45	1.09	1.28	9.77
50	1.08	1.29	1.00×10^{-2}
55	1.08	1.28	1.00
60	1.06	1.23	9.77×10^{-3}
65	1.02	1.15	9.32
70	9.62×10^{-2}	1.05	8.64
75	8.70	0.914	7.77
80	7.46	0.760	6.72
85	5.91	0.587	5.52
90	4.10	0.399	4.20
95	2.10	0.202	2.79
100	0	0	1.33

Failure Deflection and Strain Data for Minimum Failure Value of P_1/P_E

$S = 100$ $T = 0.598$

$\beta = 2$ $\Delta T = 3.93 \times 10^{-4}$

$P_1/P_E \doteq 5.10$ $n = 20$

Per cent of length from loaded end (%)	Longitudinal deflection (u_i/ℓ)	Lateral deflection (v_i/ℓ)	Extreme fiber strain
0	2.12×10^{-1}	0	0
5	1.91	0.202	0.838×10^{-3}
10	1.72	0.399	2.48
15	1.54	0.586	4.02
20	1.40	0.757	5.37
25	1.28	0.909	6.51
30	1.19	1.04	7.46
35	1.13	1.14	8.24
40	1.10	1.22	8.90
45	1.08	1.27	9.43
50	1.08	1.29	9.81
55	1.07	1.27	1.00×10^{-2}
60	1.05	1.23	9.96×10^{-3}
65	1.02	1.15	9.60
70	9.56×10^{-2}	1.04	8.89
75	8.62	0.909	7.86
80	7.36	0.754	6.61
85	5.81	0.581	5.27
90	4.01	0.395	3.90
95	2.05	0.200	2.56
100	0	0	1.25

Failure Deflection and Strain Data for Minimum Failure

Value of P_1/P_E

$S = 100$ $T = 0.109$

$\beta = 5$ $\Delta T = 3.93 \times 10^{-4}$

$P_1/P_E = 7.25$ $n = 20$

Per cent of length from loaded end (%)	Longitudinal deflection (u_i/ℓ)	Lateral deflection (v_i/ℓ)	Extreme fiber strain
0	1.63×10^{-1}	0	0
5	1.56	1.23×10^{-2}	7.93×10^{-3}
10	1.49	2.11	8.45
15	1.41	2.53	8.75
20	1.34	2.49	8.77
25	1.26	2.06	8.56
30	1.18	1.38	8.16
35	1.11	6.39×10^{-3}	7.88
40	1.03	0.114	8.40
45	9.47×10^{-2}	-3.84	8.76
50	8.66	-4.88	8.96
55	7.84	-2.99	9.01
60	7.00	1.34	8.91
65	6.16	7.17	8.72
70	5.30	1.33×10^{-2}	8.64
75	4.43	1.86	9.10
80	3.56	2.19	9.52
85	2.68	2.20	9.84
90	1.79	1.84	1.00×10^{-2}
95	0.897	1.09	9.99×10^{-3}
100	0	0	9.75

Failure Deflection and Strain Data for Minimum Failure Value of P_1/P_E

$S = 100$

$\beta = 7$

$P_1/P_E \doteq 7.50$

$T = 4.79 \times 10^{-2}$

$\Delta T = 3.93 \times 10^{-4}$

$n = 20$

Per cent of length from loaded end (%)	Longitudinal deflection (u_i/ℓ)	Lateral deflection (v_i/ℓ)	Extreme fiber strain
0	1.61×10^{-1}	0	0
5	1.54	3.04×10^{-3}	7.00×10^{-3}
10	1.48	4.51	7.13
15	1.41	4.67	7.20
20	1.34	3.87	7.25
25	1.27	2.59	7.29
30	1.19	1.27	7.42
35	1.12	2.59×10^{-4}	7.64
40	1.04	− 3.48	7.79
45	9.61×10^{-2}	− 6.36	7.93
50	8.81	− 7.44	8.06
55	7.99	− 7.80	8.20
60	7.16	− 7.28	8.36
65	8.32	− 4.67	8.53
70	5.46	1.53	8.70
75	4.59	1.21×10^{-3}	8.85
80	3.70	2.57	8.93
85	2.79	3.81	9.20
90	1.87	4.27	9.55
95	9.31×10^{-3}	3.20	9.86
100	0	0	1.00×10^{-2}

Failure Deflection and Strain Data for Minimum Failure Value of P_1/P_E

$S = 100$

$T = 4.73 \times 10^{-2}$

$\beta = 10$

$\Delta T = 1.96 \times 10^{-4}$

$P_1/P_E \doteq 5.75$

$n = 20$

Per cent of length from loaded end (%)	Longitudinal deflection (u_i/ℓ)	Lateral deflection (v_i/ℓ)	Extreme fiber strain
0	1.54×10^{-1}	0	0
5	1.49	2.90×10^{-3}	6.21×10^{-3}
10	1.43	4.39	6.41
15	1.36	4.62	6.55
20	1.30	3.90	6.66
25	1.23	2.64	6.78
30	1.16	1.31	6.93
35	1.09	2.65×10^{-4}	7.20
40	1.02	-3.62	7.41
45	9.46×10^{-2}	-6.51	7.58
50	8.68	-7.49	7.75
55	7.89	-7.81	7.94
60	7.08	-7.37	8.14
65	6.26	-4.89	8.33
70	5.42	1.28	8.54
75	4.56	1.21×10^{-3}	8.71
80	3.68	2.61	8.83
85	2.78	3.89	9.14
90	1.86	4.36	9.52
95	0.934	3.26	9.84
100	0	0	1.00×10^{-2}

Failure Deflection and Strain Data for Minimum Failure Value of P_1/P_E

$S = 100$

$T = 0.488$

$\beta = 15$

$\Delta T = 1.31 \times 10^{-4}$

$P_1/P_E \doteq 4.40$

$n = 20$

Per cent of length from loaded end (%)	Longitudinal deflection (u_i/ℓ)	Lateral deflection (v_i/ℓ)	Extreme fiber strain
0	7.86×10^{-2}	0	0
5	7.83	-1.79×10^{-2}	6.75×10^{-4}
10	7.77	-3.22	1.93×10^{-3}
15	7.68	-4.04	2.91
20	7.55	-4.11	3.48
25	7.38	-3.47	3.73
30	7.15	-2.22	3.79
35	6.87	-0.503	3.63
40	6.55	1.43	3.19
45	6.19	3.25	4.47
50	5.79	4.52	6.15
55	5.35	4.86	7.48
60	4.88	4.04	8.02
65	4.35	2.18	7.56
70	3.78	-0.288	6.28
75	3.18	-2.74	6.72
80	2.57	-4.51	8.73
85	1.96	-5.11	9.96
90	1.32	-4.35	1.00×10^{-2}
95	0.669	-2.46	8.91×10^{-3}
100	0	0	7.01

Failure Deflection and Strain Data for Minimum Failure Value of P_1/P_E

$S = 100$ $T = 0.478$

$\beta = 20$ $\Delta T = 9.82 \times 10^{-5}$

$P_1/P_E \doteq 3.50$ $n = 20$

Per cent of length from loaded end (%)	Longitudinal deflection (u_i/ℓ)	Lateral deflection (v_i/ℓ)	Extreme fiber strain
0	8.98×10^{-2}	0	0
5	8.93	-1.45×10^{-2}	7.31×10^{-4}
10	8.83	-2.64	2.05×10^{-3}
15	8.67	-3.37	3.07
20	8.46	-3.54	3.82
25	8.19	-3.15	4.32
30	7.87	-2.23	4.60
35	7.50	-0.851	4.70
40	7.09	0.816	4.44
45	6.64	2.49	4.94
50	6.17	3.75	6.44
55	5.68	4.20	7.73
60	5.16	3.57	8.31
65	4.60	1.93	7.96
70	4.00	-0.355	6.73
75	3.36	-2.63	7.09
80	2.72	-4.23	9.00
85	2.07	-4.72	1.00×10^{-2}
90	1.40	-3.97	9.97×10^{-3}
95	0.712	-2.23	9.00
100	0	0	7.34

Failure Deflection and Strain Data for Minimum Failure Value of P_1/P_E

$S = 100$ $T = 0.469$

$\beta = 30$ $\Delta T = 6.54 \times 10^{-5}$

$P_1/P_E \doteqdot 3.80$ $n = 20$

Per cent of length from loaded end (%)	Longitudinal deflection (u_i/ℓ)	Lateral deflection (v_i/ℓ)	Extreme fiber strain
0	8.99×10^{-2}	0	0
5	8.96	-1.20×10^{-2}	5.82×10^{-4}
10	8.86	−2.21	1.67×10^{-3}
15	8.72	−2.89	2.58
20	8.53	−3.14	3.23
25	8.30	−2.92	3.72
30	8.02	−2.24	4.16
35	7.68	−1.10	4.50
40	7.30	0.408	4.46
45	6.85	2.01	4.60
50	6.36	3.30	6.41
55	5.84	3.84	7.91
60	5.27	3.36	8.68
65	4.67	1.86	8.43
70	4.02	−0.282	7.28
75	3.35	−2.45	7.31
80	2.69	−4.01	9.02
85	2.03	−4.49	1.00×10^{-2}
90	1.36	−3.77	9.87×10^{-3}
95	0.681	−2.10	8.75
100	0	0	7.00

Failure Deflection and Strain Data for Minimum Failure

Value of P_1/P_E

$S = 100$ $T = 0.467$

$\beta = 40$ $\Delta T = 4.91 \times 10^{-5}$

$P_1/P_E \doteqdot 4.10$ $n. = 20$

Per cent of length from loaded end (%)	Longitudinal deflection (u_i/ℓ)	Lateral deflection (v_i/ℓ)	Extreme fiber strain
0	8.48×10^{-2}	0	0
5	8.46	-0.655×10^{-2}	2.97×10^{-4}
10	8.41	-1.21	8.66
15	8.31	-1.62	1.45×10^{-3}
20	8.17	-1.86	2.04
25	7.98	-1.88	2.63
30	7.76	-1.60	3.15
35	7.50	-0.981	3.61
40	7.21	-3.10×10^{-4}	3.65
45	6.87	1.10×10^{-2}	3.36
50	6.49	2.09	4.77
55	6.06	2.55	6.21
60	5.59	2.27	6.85
65	5.07	1.28	6.91
70	4.50	-0.163	6.38
75	3.86	-1.65	6.75
80	3.17	-2.73	8.62
85	2.43	-3.08	9.70
90	1.65	-2.62	1.00×10^{-2}
95	0.841	-1.47	9.56×10^{-3}
100	0	0	8.47

Failure Deflection and Strain Data for Minimum Failure Value of P_1/P_E

$S = 100$ $T = 8.48 \times 10^{-2}$

$\beta = 80$ $\Delta T = 2.45 \times 10^{-5}$

$P_1/P_E = 4.60$ $n = 20$

Per cent of length from loaded end (%)	Longitudinal deflection (u_i/ℓ)	Lateral deflection (v_i/ℓ)	Extreme fiber strain
0	4.56×10^{-2}	0	0
5	4.59	-0.738×10^{-3}	3.87×10^{-4}
10	4.63	-1.04	4.92
15	4.63	-0.628	2.22
20	4.60	-3.36×10^{-5}	3.61
25	4.57	0.148×10^{-3}	3.97
30	4.56	0.103	1.21
35	4.57	0.403	3.22
40	4.59	1.57	3.77
45	4.64	3.25	4.54
50	4.66	4.33	5.49
55	4.68	3.95	5.65
60	4.62	2.39	7.45
65	4.46	0.820	1.67×10^{-3}
70	4.18	-0.323	2.84
75	3.78	-1.32	4.10
80	3.27	-2.03	5.20
85	2.63	-2.09	6.70
90	1.85	-1.33	7.93
95	0.950	-0.165	9.10
100	0	0	1.00×10^{-2}

Failure Deflection and Strain Data for Minimum Failure

Value of P_1/P_E

$S = 150$

$T = 5.91$

$\beta = 0.1$

$\Delta T = 2.62 \times 10^{-4}$

$P_1/P_E \doteq 1.04$

$n = 20$

Per cent of length from loaded end (%)	Longitudinal deflection (u_i/ℓ)	Lateral deflection (v_i/ℓ)	Extreme fiber strain
0	4.80×10^{-1}	0	0
5	4.22	0.306	1.22×10^{-3}
10	3.87	0.605	2.71
15	3.47	0.889	4.14
20	3.12	1.15	5.48
25	2.84	1.38	6.69
30	2.64	1.58	7.75
35	2.51	1.74	8.63
40	2.44	1.86	9.31
45	2.41	1.93	9.77
50	2.40	1.96	1.00×10^{-2}
55	2.39	1.93	1.00
60	2.36	1.86	9.76×10^{-3}
65	2.29	1.74	9.29
70	2.15	1.58	8.61
75	1.95	1.38	7.73
80	1.68	1.15	6.67
85	1.33	0.888	5.46
90	9.23×10^{-2}	0.605	4.12
95	4.73	0.306	2.70
100	0	0	1.22

Failure Deflection and Strain Data for Minimum Failure Value of P_1/P_E

$S = 150$ $T = 3.31$

$\beta = 0.2$ $\Delta T = 2.62 \times 10^{-4}$

$P_1/P_E = 1.12$ $n = 20$

Per cent of length from loaded end (%)	Longitudinal deflection (u_i/ℓ)	Lateral deflection (v_i/ℓ)	Extreme fiber strain
0	4.81×10^{-1}	0	0
5	4.33	0.307	1.21×10^{-3}
10	3.88	0.606	2.70
15	3.47	0.890	4.14
20	3.12	1.15	5.48
25	2.85	1.39	6.69
30	2.64	1.58	7.75
35	2.51	1.75	8.63
40	2.44	1.86	9.31
45	2.41	1.93	9.77
50	2.40	1.96	1.00×10^{-2}
55	2.39	1.93	1.00
60	2.36	1.86	9.76×10^{-3}
65	2.29	1.74	9.29
70	2.16	1.58	8.61
75	1.95	1.38	7.73
80	1.68	1.15	6.67
85	1.33	0.889	5.45
90	9.24×10^{-2}	0.605	4.12
95	4.74	0.306	2.69
100	0	0	1.21

Failure Deflection and Strain Data for Minimum Failure Value of P_1/P_E

$S = 150$

$T = 1.73$

$\beta = 0.5$

$\Delta T = 2.62 \times 10^{-4}$

$P_1/P_E \doteq 1.46$

$n = 20$

Per cent of length from loaded end (%)	Longitudinal deflection (u_i/ℓ)	Lateral deflection (v_i/ℓ)	Extreme fiber strain
0	4.89×10^{-1}	0	0
5	4.41	0.308	1.02×10^{-3}
10	3.96	0.609	2.48
15	3.55	0.895	3.90
20	3.20	1.16	5.25
25	2.91	1.40	6.48
30	2.71	1.60	7.56
35	2.57	1.76	8.48
40	2.50	1.88	9.20
45	2.46	1.96	9.70
50	2.46	1.98	9.97
55	2.45	1.96	1.00×10^{-2}
60	2.43	1.89	9.79×10^{-3}
65	2.35	1.77	9.35
70	2.22	1.61	8.68
75	2.01	1.41	7.79
80	1.73	1.17	6.72
85	1.37	0.904	5.48
90	9.54×10^{-2}	0.616	4.10
95	4.89	0.312	2.63
100	0	0	1.10

Failure Deflection and Strain Data for Minimum Failure Value of P_1/P_E

$S = 150$ $T = 0.966$

$\beta = 1$ $\Delta T = 2.62 \times 10^{-4}$

$P_1/P_E \doteqdot 2.65$ $n = 20$

Per cent of length from loaded end (%)	Longitudinal deflection (v_i/ℓ)	Lateral deflection (v_i/ℓ)	Extreme fiber strain
0	5.02×10^{-1}	0	0
5	4.53	0.315	0.896×10^{-3}
10	4.05	0.622	2.41
15	3.63	0.914	3.88
20	3.26	1.18	5.27
25	2.97	1.42	6.54
30	2.76	1.63	7.65
35	2.62	1.79	8.57
40	2.55	1.92	9.28
45	2.52	1.99	9.76
50	2.51	2.01	1.00×10^{-2}
55	2.51	1.99	1.00
60	2.48	1.92	9.76×10^{-3}
65	2.41	1.79	9.28
70	2.27	1.63	8.58
75	2.06	1.42	7.67
80	1.76	1.18	6.58
85	1.40	0.915	5.32
90	9.72×10^{-2}	0.623	3.94
95	4.98	0.315	2.46
100	0	0	0.938

Failure Deflection and Strain Data for Minimum Failure Value of P_1/P_E

$S = 150$ $T = 0.601$

$\beta = 2$ $\Delta T = 2.62 \times 10^{-4}$

$P_1/P_E = 6.60$ $n = 20$

Per cent of length from loaded end (%)	Longitudinal deflection (u_i/ℓ)	Lateral deflection (v_i/ℓ)	Extreme fiber strain
0	4.88×10^{-1}	0	0
5	4.41	0.307	0.742×10^{-3}
10	3.96	0.606	2.23
15	3.55	0.891	3.68
20	3.20	1.15	5.02
25	2.93	1.39	6.20
30	2.72	1.59	7.24
35	2.59	1.76	8.14
40	2.51	1.88	8.91
45	2.48	1.95	9.51
50	2.47	1.98	9.89
55	2.47	1.96	1.00×10^{-2}
60	2.44	1.89	9.84×10^{-3}
65	2.37	1.78	9.39
70	2.24	1.61	8.70
75	2.03	1.41	7.83
80	1.75	1.18	6.82
85	1.39	0.911	5.66
90	9.72×10^{-2}	0.621	4.34
95	4.99	0.315	2.83
100	0	0	1.19

Failure Deflection and Strain Data for Minimum Failure Value of P_1/P_E

$S = 150$

$T = 0.153$

$\beta = 5$

$\Delta T = 2.62 \times 10^{-4}$

$P_1/P_E \doteq 11.4$

$n = 20$

Per cent of length from loaded end (%)	Longitudinal deflection (u_i/ℓ)	Lateral deflection (v_i/ℓ)	Extreme fiber strain
0	9.86×10^{-2}	0	0
5	9.23	0.752×10^{-1}	5.12×10^{-3}
10	8.71	1.35	7.77
15	8.31	1.67	9.51
20	7.95	1.66	9.97
25	7.54	1.35	9.05
30	7.02	0.797	6.97
35	6.44	0.149	4.23
40	5.38	− 0.450	6.05
45	5.42	− 0.868	8.32
50	5.02	− 1.01	9.53
55	4.63	− 0.862	9.47
60	4.15	− 0.447	8.19
65	3.59	0.136	5.99
70	3.01	0.760	4.48
75	2.48	1.29	7.09
80	2.04	1.59	9.08
85	1.65	1.60	1.00×10^{-2}
90	1.21	1.30	9.70×10^{-3}
95	0.657	0.726	8.14
100	0	0	5.65

Failure Deflection and Strain Data for Minimum Failure Value of P_1/P_E

$S = 150$ $\qquad T = 0.113$

$\beta = 7$ $\qquad \Delta T = 1.87 \times 10^{-4}$

$P_1/P_E \doteqdot 14.0$ $\qquad n. = 20$

Per cent of length from loaded end (%)	Longitudinal deflection (u_i/ℓ)	Lateral deflection (v_i/ℓ)	Extreme fiber strain
0	1.06×10^{-1}	0	0
5	1.00	6.21×10^{-2}	5.27×10^{-3}
10	9.53×10^{-2}	1.11×10^{-1}	7.46
15	9.10	1.37	8.96
20	8.70	1.36	9.45
25	8.25	1.08	8.76
30	7.72	6.01×10^{-2}	7.05
35	7.13	0.315	4.72
40	6.56	−4.98	6.31
45	6.05	−8.67	8.39
50	5.60	−9.96	9.54
55	5.13	−8.61	9.53
60	4.60	−4.96	8.42
65	3.99	0.166	6.51
70	3.37	5.63	5.27
75	2.78	1.02×10^{-1}	7.51
80	2.26	1.30	9.21
85	1.78	1.32	1.00×10^{-2}
90	1.27	1.07	9.76×10^{-3}
95	0.670	6.06×10^{-2}	8.48
100	0	0	6.41

Failure Deflection and Strain Data for Minimum Failure Value of P_1/P_E

$S = 150$

$T = 7.23 \times 10^{-2}$

$\beta = 10$

$\Delta T = 1.31 \times 10^{-4}$

$P_1/P_E = 16.75$

$n = 20$

Per cent of length from loaded end (%)	Longitudinal deflection (u_i/ℓ)	Lateral deflection (v_i/ℓ)	Extreme fiber strain
0	1.43×10^{-1}	0	0
5	1.37	2.47×10^{-2}	6.57×10^{-3}
10	1.31	4.27	7.47
15	1.25	5.10	8.00
20	1.19	4.88	8.10
25	1.13	3.77	7.82
30	1.06	2.07	7.28
35	9.94×10^{-2}	0.159	6.65
40	9.25	− 1.59	7.39
45	8.56	− 2.83	8.12
50	7.86	− 3.30	8.62
55	7.14	− 2.92	8.80
60	6.40	− 1.75	8.63
65	5.64	-4.58×10^{-4}	8.19
70	4.86	1.84×10^{-2}	7.67
75	4.06	3.51	8.56
80	3.27	4.62	9.31
85	2.47	4.88	9.82
90	1.66	4.15	1.00×10^{-2}
95	0.840	2.45	9.80×10^{-3}
100	0	0	9.15

Failure Deflection and Strain Data for Minimum Failure Value of P_1/P_E

$S = 150$

$T = 3.10 \times 10^{-2}$

$\beta = 15$

$\Delta T = 8.73 \times 10^{-5}$

$P_1/P_E \doteq 13.0$

$n = 20$

Per cent of length from loaded end (%)	Longitudinal deflection (u_i/ℓ)	Lateral deflection (v_i/ℓ)	Extreme fiber strain
0	1.55×10^{-1}	0	0
5	1.49	3.69×10^{-3}	6.25×10^{-3}
10	1.43	5.01	6.46
15	1.37	4.40	6.53
20	1.30	2.72	6.58
25	1.24	8.91×10^{-4}	6.75
30	1.17	-3.88	7.05
35	1.10	-8.74	7.24
40	1.02	-7.36	7.40
45	9.48×10^{-2}	-3.12	7.58
50	8.71	1.17	7.76
55	7.91	3.37	8.00
60	7.10	1.52	8.17
65	6.28	-3.71	8.28
70	5.43	-7.48	8.54
75	4.57	-4.26	8.83
80	3.68	8.89	9.03
85	2.78	2.90×10^{-3}	9.07
90	1.86	4.49	9.43
95	0.931	3.99	9.82
100	0	0	1.00×10^{-2}

Failure Deflection and Strain Data for Minimum Failure Value of P_1/P_E

$S = 150$

$T = 3.04 \times 10^{-2}$

$\beta = 20$

$\Delta T = 6.54 \times 10^{-5}$

$P_1/P_E \doteq 11.2$

$n = 20$

Per cent of length from loaded end (%)	Longitudinal deflection (u_i/ℓ)	Lateral deflection (v_i/ℓ)	Extreme fiber strain
0	1.48×10^{-1}	0	0
5	1.43	3.52×10^{-3}	5.16×10^{-3}
10	1.38	4.94	5.50
15	1.32	4.47	5.72
20	1.27	2.84	5.88
25	1.21	9.55×10^{-4}	6.04
30	1.14	-4.09	6.45
35	1.08	-9.35	6.73
40	1.01	-7.76	6.97
45	9.38×10^{-2}	-3.10	7.20
50	8.64	1.34	7.45
55	7.87	3.61	7.70
60	7.09	1.84	7.95
65	6.27	-3.67	8.16
70	5.44	-7.97	8.47
75	4.57	-5.03	8.85
80	3.68	8.59	9.07
85	2.77	3.01×10^{-3}	9.09
90	1.85	4.71	9.46
95	0.930	4.18	9.84
100	0	0	1.00×10^{-2}

Failure Deflection and Strain Data for Minimum Failure Value of P_1/P_E

$S = 150$

$T = 0.404$

$\beta = 30$

$\Delta T = 4.36 \times 10^{-5}$

$P_1/P_E \doteqdot 8.40$

$n = 20$

Per cent of length from loaded end (%)	Longitudinal deflection (u_i/ℓ)	Lateral deflection (v_i/ℓ)	Extreme fiber strain
0	9.79×10^{-2}	0	0
5	9.75	-1.05×10^{-2}	0.582×10^{-3}
10	9.65	−1.82	1.77
15	9.50	−2.03	2.77
20	9.28	−1.61	3.33
25	8.99	−0.698	3.59
30	8.64	0.480	3.53
35	8.25	1.56	4.48
40	7.82	2.11	5.66
45	7.34	1.88	6.44
50	6.81	0.803	6.58
55	6.23	−0.778	5.95
60	5.63	−2.20	6.89
65	5.01	−2.80	8.18
70	4.36	−2.24	8.67
75	3.67	−0.669	8.25
80	2.95	1.33	6.96
85	2.23	2.90	8.69
90	1.50	3.31	9.92
95	0.759	2.24	1.00×10^{-2}
100	0	0	8.78×10^{-3}

Failure Deflection and Strain Data for Minimum Failure

Value of P_1/P_E

$S = 150$ $T = 0.402$

$\beta = 40$ $\Delta T = 3.27 \times 10^{-5}$

$P_1/P_E \doteq 8.30$ $n = 20$

Per cent of length from loaded end (%)	Longitudinal deflection (u_i/ℓ)	Lateral deflection (v_i/ℓ)	Extreme fiber strain
0	9.64×10^{-2}	0	0
5	9.60	-3.80×10^{-4}	0.569×10^{-3}
10	9.50	-0.283×10^{-2}	1.48
15	9.32	−0.741	1.87
20	9.06	−1.08	3.09
25	8.74	−0.999	4.14
30	8.35	−0.434	4.73
35	7.90	0.519	4.78
40	7.42	1.44	5.51
45	6.91	1.66	6.82
50	6.38	0.873	6.96
55	5.82	−0.572	6.20
60	5.24	−2.05	6.29
65	4.67	−2.83	7.66
70	4.08	−2.38	8.25
75	3.45	−0.804	7.79
80	2.79	1.29	6.52
85	2.12	3.06	8.09
90	1.44	3.60	9.75
95	0.733	2.44	1.00×10^{-2}
100	0	0	8.57×10^{-3}

Failure Deflection and Strain Data for Minimum Failure Value of P_1/P_E

$S = 150$ | $T = 0.289$

$\beta = 80$ | $\Delta T = 1.64 \times 10^{-5}$

$P_1/P_E \doteq 9.8$ | $n = 20$

Per cent of length from loaded end (%)	Longitudinal deflection (u_i/ℓ)	Lateral deflection (v_i/ℓ)	Extreme fiber strain
0	-6.34×10^{-2}	0	0
5	-6.36	-4.19×10^{-3}	0.386×10^{-3}
10	-6.40	-6.01	0.921
15	-6.49	-5.90	1.11
20	-6.57	-4.67	1.12
25	-6.62	-1.60	0.803
30	-6.64	2.81	0.192
35	-6.62	5.93	0.502
40	-6.53	6.95	1.19
45	-6.38	7.26	1.73
50	-6.16	6.22	2.48
55	-5.87	3.38	2.91
60	-5.53	2.03	3.83
65	-5.11	3.21	4.31
70	-4.61	3.12	5.48
75	-4.02	0.109	6.20
80	-3.37	-3.36	6.65
85	-2.64	-5.53	7.56
90	-1.82	-6.29	8.51
95	-0.930	-4.82	9.48
100	0	0	1.00×10^{-2}

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